arXiv:hep-th/9505056v1 10 May 1995

May 1995 DFUB 95–3
Version 1 hep-th/9505056

EXTRINSIC HERMITIAN GEOMETRY OF FUNCTIONAL DETERMINANTS FOR VECTOR SUBBUNDLES AND THE DRINFELD–SOKOLOV GHOST SYSTEM

by

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Abstract

In this paper, a novel method is presented for the study of the dependence of the functional determinant of the Laplace operator associated to a subbundle F of a hermitian holomorphic vector bundle E over a Riemann surface Σ on the hermitian structure (h, H) of E. The generalized Weyl anomaly of the effective action is computed and found to be expressible in terms of a suitable generalization of the Liouville and Donaldson actions. The general techniques worked out are then applied to the study of a specific model, the Drinfeld–Sokolov (DS) ghost system arising in W–gravity. The expression of generalized Weyl anomaly of the DS ghost effective action is found. It is shown that, by a specific choice of the fiber metric H_h depending on the base metric h, the effective action reduces into that of a conformal field theory. Its central charge is computed and found to agree with that obtained by the methods of hamiltonian reduction and conformal field theory. The DS holomorphic gauge group and the DS moduli space are defined and their dimensions are computed.

1. Introduction

In the last thirty years, a large body of physical literature has been devoted to the study of functional determinants in connection with quantum gravity, gauge theory and, more recently, string theory. Several methods for their computation have been developed such as zeta function regularization [1–5], proper time regularization [6] and Fujikawa's method [7] to mention only the most frequently used. All these approaches analyze the dependence of the determinants on the relevant background fields and employ in a crucial manner the Seeley–De Witt coefficients of the associated heat kernels [8–10].

In this paper, a novel method is presented for the analysis of functional determinants of Laplace operators associated to a subbundle of a holomorphic vector bundle on a Riemann surface from an extrinsic point of view. The general techniques worked out are then applied to the study of a specific model, the Drinfeld–Sokolov ghost system arising in W–gravity. The results obtained in this way are interesting both as illustration of the general formalism and for its applications to W–strings.

The problem tackled in the first part of this paper can be stated as follows. Consider a holomorphic vector bundle E on a Riemann surface Σ and a subbundle F of E. A hermitian structure (h, H) on E is a pair consisting of a hermitian metric h on Σ and a hermitian fiber metric H. When E is equipped with a hermitian structure (h, H), a hermitian structure (h, H_F) is induced on the subbundle F. This allows to construct the Laplace operator $\Delta_{w,F;h,H} = \bar\partial^\star_{w,F;h,H}\bar\partial_{w,F;h,H}$ associated to the Cauchy–Riemann operator $\bar\partial_{w,F}$ acting on F–valued conformal fields of weight w. Using proper time regularization, one may then define the determinant $\det'_\epsilon\Delta_{w,F;h,H}$, where ϵ is the proper time ultraviolet cut–off. Two different approaches to the study of such determinant can be envisaged. In the 'intrinsic' approach, one considers F as a holomorphic vector bundle on its own right equipped with the induced hermitian structure (h, H_F). The problem is then reduced to the standard one of studying $\det'_\epsilon\Delta_{w,E;h,H}$ for a holomorphic vector bundle E endowed with a hermitian structure (h, H) [11]. This approach has the drawback that all results are expressed in terms of H_F, which depends in a complicated way on H, while, in certain physical applications, one would like to express the results directly in terms of H. Hence, an 'extrinsic' approach capable of computing the dependence of $\det'_\epsilon\Delta_{w,F;h,H}$ on the hermitian structure (h, H) of E would be desirable.

In sect. 2, $\det'_\epsilon\Delta_{w,F;h,H}$ is studied as a functional of (h, H) in the framework of the Liouville–Donaldson parametrization of the family of hermitian structures [12]. The expressions obtained involve the H–hermitian fiber projector $\varpi_{F;H}$ of E onto F, which is a

local functional of H. In sect. 3, the class of special holomorphic structures of the smooth vector bundle E, for which the smooth subbundle F is holomorphic, is characterized in the framework of the Beltrami–Koszul parametrization of the holomorphic structures [12]. Further, it is shown that the special subgroup of the automorphism group of E preserving F preserves such class of holomorphic structures and is the symmetry group under which $\det'_\epsilon \Delta_{w,E;h,H}$ is invariant.

In the second part of the paper, the results outlined above are applied to the study of the renormalized effective action of the Drinfeld–Sokolov ghost system in W–gravity [13–15]. Let us briefly recall the formulation of the model. Let G be a simple complex Lie group and let S be an $SL(2,\mathbb{C})$ subgroup of G invariant under the compact conjugation $\dagger$ of G. To these algebraic data, there is associated a halfinteger grading of $\mathfrak{g}$ and a certain bilinear form χ on $\mathfrak{g}$ [14]. On a Riemann surface Σ with a spinor structure $k^{\otimes\frac{1}{2}}$, one can further associate to the pair (G,S) a holomorphic G–valued cocycle defining holomorphic principal G–bundle, the Drinfeld–Sokolov (DS) bundle L [16]. $\mathrm{Ad}\,L$ is then a holomorphic vector bundle. If $\mathfrak{x}$ is a maximal negative graded subalgebra of $\mathfrak{g}$ isotropic with respect to χ, then the $\mathfrak{x}$–valued sections of $\mathrm{Ad}\,L$ span a holomorphic subbundle $\mathrm{Ad}\,L_{\mathfrak{x}}$ of $\mathrm{Ad}\,L$.

W gravity may be formulated as a gauge theory based on the smooth principal G bundle underlying the DS bundle L. The gauge fields are $\mathfrak{x}$–valued sections of $\bar{k}\otimes \mathrm{Ad}\,L$. The gauge group, the DS gauge group Gau_{cDS}, consists of the $\exp\mathfrak{x}$–valued gauge transformations. Fixing such gauge symmetry yields the DS ghosts β, γ as Fadeev–Popov ghosts. Here, β is an anticommuting section of $k\otimes \mathrm{Ad}\,L$ valued in $\mathfrak{g}/\mathfrak{x}^\perp$, where $\mathfrak{x}^\perp$ is the orthogonal complement of $\mathfrak{x}$ with respect to the Cartan Killing form $\mathrm{tr}_{\,\mathrm{ad}}$ of $\mathfrak{g}$. γ is an anticommuting $\mathfrak{x}$–valued section of $\mathrm{Ad}\,L$. The action is given by

$$S_{DS}(\beta,\beta^\dagger,\gamma,\gamma^\dagger) = \frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}_{\,\mathrm{ad}}\big(\beta\bar{\partial}\gamma\big) + \mathrm{c.\ c.}. \tag{1.1}$$

The quantum effective action is thus related to the determinant $\det'_\epsilon \Delta_{0,\,\mathrm{Ad}\,L_{\mathfrak{x}};h,\,\mathrm{Ad}\,H}$ relative to the holomorphic vector subbundle $\mathrm{Ad}\,L_{\mathfrak{x}}$ of $\mathrm{Ad}\,L$, where H is a hermitian metric on L. In sect. 4, the effective action is studied by means of the general methods developed in sects. 3 and 4. It is shown that the metric h can be lifted to a metric H_h on L depending only h. Setting $H = H_h$ in the determinant, one finds that the resulting renormalized effective action is that of a conformal field theory perhaps perturbed by a term of the form $\int \sqrt{h}{R_h}^2$ as in the model considered in refs. [18–19]. Its central charge is computed and found to agree with that obtained by the methods of hamiltonian reduction and conformal field theory [14]. The dimensions of the spaces of β– and γ–zero modes and the index

of the ghost kinetic operator are also computed. Finally, the relevant classes of special holomorphic structures and special automorphisms of $\mathrm{Ad}\,L$ are defined and studied. A notion of stability for holomorphic structures is introduced. The holomorphic DS gauge group and of the DS moduli space of stable holomorphic structures are then defined and their dimensions computed. It must be emphasized that the above is the DS moduli space and is distinct from the W–moduli space introduced by Hitchin in ref. [20] and later identified with the moduli space of quantum W–gravity in ref. [21].

2. The determinant of $\Delta^\sharp_{w,F;h,H}$

In what follows, E is a holomorphic vector bundle of rank r_E over a compact connected Riemann surface Σ of genus ℓ. $k^{\otimes\frac{1}{2}}$ is a fixed tensor square root of the canonical line bundle k of Σ, *i. e.* a spinor structure in physical parlance. F is a holomorphic vector subbundle of E of rank $r_F > 0$. See ref. [17] for basic background.

Let $w,\ \bar w \in \mathbb{Z}/2$. Denote by $\mathcal{S}_{w,\bar w}$ the complex vector space of smooth sections of the complex line bundle $k^{\otimes w} \otimes \bar k^{\otimes \bar w}$. The elements of $\mathcal{S}_{w,\bar w}$ are ordinary conformal fields of weights $w,\ \bar w$. If V is a smooth vector bundle related to either E or F, denote by $\mathcal{S}_{w,\bar w,V}$ the complex vector space of smooth sections of the complex vector bundle $k^{\otimes w}\otimes \bar k^{\otimes \bar w}\otimes V$. The elements of $\mathcal{S}_{w,\bar w,V}$ are generalized vector valued conformal fields of weights $w,\ \bar w$.

A hermitian structure (h,H) on E consists of a hermitian metric h on the base Σ and a hermitian fiber metric H, *i. e.* a section h of $k\otimes\bar k$ such that $h>0$ and a section H of $E\otimes \bar E$ such that $H = H^\dagger > 0$. To any hermitian structure (h,H) on E there is associated a Hilbert inner product on $\mathcal{S}_{w,\bar w,F}$ by

$$\langle \phi,\psi\rangle_{w,\bar w,F;h,H} = \int_\Sigma d^2 z h^{1-w-\bar w}\phi^\dagger H^{-1}\psi, \quad \phi,\ \psi\in\mathcal{S}_{w,\bar w,F}. \tag{2.1}$$

By completing $\mathcal{S}_{w,\bar w,F}$ with respect to the corresponding norm, one obtains a complex Hilbert space $\mathcal{H}_{w,\bar w,F;h,H}$ containing $\mathcal{S}_{w,\bar w,F}$ as a dense subspace.

The Cauchy–Riemann operator $\bar\partial_{w,F}$ is the linear operator from $\mathcal{S}_{w,0,F}$ to $\mathcal{S}_{w,1,F}$ locally given by $\bar\partial_{w,F} = \bar\partial$ on $\mathcal{S}_{w,0,F}$. $\bar\partial_{w,F}$ can be extended to a linear operator from a dense subspace of $\mathcal{H}_{w,0,F;h,H}$ containing $\mathcal{S}_{w,0,F}$ into $\mathcal{H}_{w,1,F;h,H}$. Its adjoint $\bar\partial^\star_{w,F;h,H}$ is a linear operator from a dense subspace of $\mathcal{H}_{w,1,F;h,H}$ containing $\mathcal{S}_{w,1,F}$ into $\mathcal{H}_{w,0,F;h,H}$. Using $\bar\partial_{w,F;h,H}$ and $\bar\partial^\star_{w,F;h,H}$, one can define the Laplace operators

$$\Delta_{w,F;h,H} = \bar\partial^\star_{w,F;h,H}\bar\partial_{w,F;h,H}, \tag{2.2}$$

$$\Delta^\vee_{w,F;h,H} = \bar\partial_{w,F;h,H}\bar\partial^\star_{w,F;h,H}. \tag{2.3}$$

$\Delta_{w,F;h,H}$ is a linear operator from a dense subspace of $\mathcal{H}_{w,0,F;h,H}$ containing $\mathcal{S}_{w,0,F}$ into $\mathcal{H}_{w,0,F;h,H}$. $\Delta^{\vee}_{w,F;h,H}$ is a linear operator from a dense subspace of $\mathcal{H}_{w,1,F;h,H}$ containing $\mathcal{S}_{w,1,F}$ into $\mathcal{H}_{w,1,F;h,H}$. $\Delta_{w,F;h,H}$ and $\Delta^{\vee}_{w,F;h,H}$ are essentially self–adjoint unbounded elliptic linear differential operators with a discrete non negative spectrum of finite multiplicity. Furthermore, $\Delta_{w,F;h,H}$ and $\Delta^{\vee}_{w,F;h,H}$ have the same spectrum and their non zero eigenvalues have the same multiplicity.

One has that $\ker\bar{\partial}_{w,F;h,H} = \ker\Delta_{w,F;h,H} \cong \ker\bar{\partial}_{w,F}$ and $\ker\bar{\partial}^{\star}_{w,F;h,H} = \ker\Delta^{\vee}_{w,F;h,H} \cong \operatorname{coker}\bar{\partial}_{w,F}$. The difference $\operatorname{ind}\bar{\partial}_{w,F} = \dim\ker\bar{\partial}_{w,F} - \dim\operatorname{coker}\bar{\partial}_{w,F}$ is the Atiyah–Singer index of $\bar{\partial}_{w,F}$ and is a topological invariant, *i. e.* it is independent from the background holomorphic structure of E and from the hermitian structure (h,H). One has thus

$$\operatorname{ind}\bar{\partial}_{w,F} = \dim\ker\bar{\partial}_{w,F;h,H} - \dim\ker\bar{\partial}^{\star}_{w,F;h,H} = \dim\ker\Delta_{w,F;h,H} - \dim\ker\Delta^{\vee}_{w,F;h,H}. \tag{2.4}$$

In field theory, the objects of main interest are the functional determinants of $\Delta_{w,F;h,H}$ and $\Delta^{\vee}_{w,F;h,H}$. In this paper, these will be defined by the proper time method [6]. In such approach, the zero eigenvalues are excluded in order to get a non trivial result. Further, since the spectrum of the operators considered is not bounded above, it is necessary to introduce an ultraviolet cut–off $1/\epsilon$ with $\epsilon > 0$. One thus uses the standard notation $\det'_\epsilon$ to denote the cut–off determinant with the zero eigenvalues removed. Since the non zero spectra of $\Delta_{w,F;h,H}$ and $\Delta^{\vee}_{w,F;h,H}$ are identical, one knows a priori that

$$\det{}'_\epsilon\Delta_{w,F;h,H} = \det{}'_\epsilon\Delta^{\vee}_{w,F;h,H}. \tag{2.5}$$

It is thus convenient to denote by $\Delta^{\sharp}_{w,F;h,H}$ either $\Delta_{w,F;h,H}$ or $\Delta^{\vee}_{w,F;h,H}$. Following [6], one has

$$\ln\det{}'_\epsilon\Delta^{\sharp}_{w,F;h,H} = -\int_\epsilon^\infty \frac{dt}{t}\Big[\operatorname{Tr}\big(\exp(-t\Delta^{\sharp}_{w,F;h,H})\big) - d^{\sharp}_{w,F}\Big], \tag{2.6}$$

where $d^{\sharp}_{w,F} = \dim\ker\Delta^{\sharp}_{w,F;h,H}$. Using the small t expansion of the diagonal part of the heat kernel $\exp(-t\Delta^{\sharp}_{w,F;h,H})$ of $\Delta^{\sharp}_{w,F;h,H}$, one can compute the terms of $\ln\det'_\epsilon\Delta^{\sharp}_{w,F;h,H}$, which diverge as $\epsilon \to 0$.

As well–known, it is very difficult to compute $\det'_\epsilon\Delta^{\sharp}_{w,F;h,H}$ as a functional of (h,H) directly from (2.6). It is instead relatively easier to compute the variation of $\det'_\epsilon\Delta^{\sharp}_{w,F;h,H}$ with respect to (h,H). To this end, one introduces the differential complex $(\delta,\Omega^*_{\mathfrak{H}_E})$ where δ is the differential operator on the infinite dimensional manifold $\mathfrak{H}_E$ of hermitian structures (h,H) on E satisfying $\delta^2 = 0$ and $\Omega^*_{\mathfrak{H}_E}$ is the corresponding exterior algebra. The cohomology ring $H^*(\delta,\Omega^*_{\mathfrak{H}_E})$ is trivial since $\mathfrak{H}_E$ is contractible.

The variation of the Hilbert space structure defined by (2.1) is given by an expression of the form

$$\delta\langle\phi,\psi\rangle_{w,\bar w,F;h,H}=\langle\phi,Q_{w,\bar w,F;h,H}\psi\rangle_{w,\bar w,F;h,H},\quad \phi,\ \psi\in\mathcal{S}_{w,\bar w,F}. \tag{2.7}$$

Here, $Q_{w,\bar w,F;h,H}$ is an element of the tensor product $\mathcal{S}_{0,0,\,\mathrm{End}\,F}\otimes\Omega^1_{\mathfrak{H}_E}$, where $\mathrm{End}\,F$ is the endomorphism bundle of F. $Q_{w,\bar w,F;h,H}$ acts as multiplicative operator in $\mathcal{H}_{w,\bar w,F;h,H}$ valued in $\Omega^1_{\mathfrak{H}_E}$ and, as such, it is bounded and self–adjoint.

One can show that there are bases $\{\omega_{w,F;h,H;i}|i=1,\ldots,d_{w,F}\}$ and $\{\omega^\vee_{w,F;h,H;i}|i=1,\ldots,d^\vee_{w,F}\}$ of $\ker\Delta_{w,F;h,H}$ and $\ker\Delta^\vee_{w,F;h,H}$, respectively, such that

$$\delta\omega_{w,F;h,H;i}=0,\quad i=1,\ldots,d_{w,F} \tag{2.8}$$

$$\delta\omega^\vee_{w,F;h,H;i}=-\,Q_{w,1,F;h,H}\omega^\vee_{w,F;h,H;i},\quad i=1,\ldots,d^\vee_{w,F} \tag{2.9}$$

The Gram matrices of these bases are

$$M_{w,F;h,H}(\omega)_{i,j}=\langle\omega_{w,F;h,H;i},\omega_{w,F;h,H;j}\rangle_{w,0,F;h,H},\quad i,\ j=1,\ldots,d_{w,F}, \tag{2.10}$$

$$M_{w,F;h,H}(\omega^\vee)_{i,j}=\langle\omega^\vee_{w,F;h,H;i},\omega^\vee_{w,F;h,H;j}\rangle_{w,1,F;h,H},\quad i,\ j=1,\ldots,d^\vee_{w,F}. \tag{2.11}$$

A standard analysis shows that

$$\begin{aligned}&\delta\ln{\det}'_\epsilon\Delta^\sharp_{w,F;h,H}=\delta\ln\det M_{w,F;h,H}(\omega)+\delta\ln\det M_{w,F;h,H}(\omega^\vee)\\&-\,\mathrm{Tr}\big(Q_{w,0,F;h,H}\exp(-\epsilon\Delta_{w,F;h,H})\big)+\,\mathrm{Tr}\big(Q_{w,1,F;h,H}\exp(-\epsilon\Delta^\vee_{w,F;h,H})\big).\end{aligned} \tag{2.12}$$

The last two traces can be dealt with by using the small t expansion of the diagonal part of the heat kernel $\exp(-t\Delta^\sharp_{w,F;h,H})$ of $\Delta^\sharp_{w,F;h,H}$. In principle, this relation can be integrated and yield an expression for ${\det}'_\epsilon\Delta^\sharp_{w,F;h,H}$ up to a constant. The part of the constant that diverges in the ultraviolet limit $\epsilon\to 0$ can be computed from (2.6).

The above method for computing functional determinants, and other methods as well, exploit heat kernel techniques in an essential way. The elliptic operators Δ considered here act on a suitable space of sections of some smooth vector bundle V and are given locally by an expression of the form

$$\Delta=-h^{-1}\big(1\bar\partial\partial+\sigma\bar\partial+\sigma^*\partial+\tau\big), \tag{2.13}$$

where σ, σ^* and τ are certain smooth matrix valued functions. A standard calculation on the same line as those described in ref. [6] yields for the diagonal part of the heat kernel $\exp(-t\Delta)$ of Δ the local expression

$$\exp(-t\Delta)_{\mathrm{diag}}=\frac{1}{\pi t}h1-\frac{1}{6\pi}\bar\partial\partial\ln h1-\frac{1}{2\pi}\big(\partial\sigma^*+\bar\partial\sigma+\sigma^*\sigma+\sigma\sigma^*-2\tau\big)+O(t). \tag{2.14}$$

By covariance, $\exp(-\Delta)_{\rm diag}$ must belong to $\mathcal{S}_{1,1,\,{\rm End}\,V}$.

One must now proceed to the implementation of the methods described above. As explained in the introduction, instead of considering the hermitian structure induced by (h,H) on F and carrying out the calculation intrinsically, one is interested in expressing the determinant directly in terms of (h,H). For such reason, one introduces the orthogonal projector $\varpi_{w,\bar w,F;h,H}$ of $\mathcal{H}_{w,\bar w,E;h,H}$ onto $\mathcal{H}_{w,\bar w,F;h,H}$. $\varpi_{w,\bar w,F;h,H}$ is a bounded self–adjoint multiplicative operator corresponding to a smooth section $\varpi_{F;H}$ of $F\otimes E^\vee$ independent from w, $\bar w$ and h, where $E^\vee$ is the dual vector bundle of E. As F is a subbundle of E, $\varpi_{F;H}$ is also an element of $\mathcal{S}_{0,0,\,{\rm End}\,E}$. The fact that $\varpi_{w,\bar w,F;h,H}$ is an orthogonal projector in $\mathcal{H}_{w,\bar w,E;h,H}$ implies that

$$\varpi_{F;H}{}^2=\varpi_{F;H} \tag{2.15}$$

$$H\varpi_{F;H}{}^\dagger H^{-1}=\varpi_{F;H}. \tag{2.16}$$

If $\phi\in\mathcal{S}_{w,0,F}$, then $\bar\partial_{w,F}\phi\in\mathcal{S}_{w,1,F}$, so that one has at the same time that $\varpi_{F;H}\phi=\phi$ and $\varpi_{F;H}\bar\partial_{w,F}\phi=\bar\partial_{w,F}\phi$. From this remark and (2.16), the following equivalent relations follow

$$\bar\partial\varpi_{F;H}\varpi_{F;H}=0,\quad \varpi_{F;H}\partial_H\varpi_{F;H}=0, \tag{2.17}$$

where $\partial_H=\partial-{\rm ad}\,(\partial HH^{-1})$ is the covariant derivative on ${\rm End}\,E$ associated to the metric connection ∂HH^{-1} [1]. Projectors $\varpi_{F;H}$ satisfying (2.17) were introduced earlier in the mathematical literature in the analysis of Hermitian–Einstein and Higgs bundles [22–23].

If $\phi\in\mathcal{S}_{w,0,F}$, then $\varpi_{F;H}\phi=\phi$, so that $\varpi_{F;H}\phi$ is independent from H. This implies the relation

$$\delta\varpi_{F;H}\varpi_{F;H}=0. \tag{2.18}$$

By combining (2.16) and (2.18), one obtains

$$\delta\varpi_{F;H}=-\varpi_{F;H}\delta HH^{-1}(1-\varpi_{F;H}). \tag{2.19}$$

This identity is of crucial importance. It is a functional differential equation constraining the dependence of $\varpi_{F;H}$ on H and shows that $\varpi_{F;H}$ is a local functional of H.

[1] By convention, each differential operator acts on the object immediately at its right.

By direct calculation, one finds that the differential operators $\bar\partial_{w,F;h,H}$, $\bar\partial^\star_{w,F;h,H}$, $\Delta_{w,F;h,H}$ and $\Delta^\vee_{w,F;h,H}$ defined earlier have the following local expressions

$$\bar\partial_{w,F;h,H} = 1\bar\partial, \quad \text{on } \mathcal{S}_{w,0,F}, \tag{2.20}$$

$$\bar\partial^\star_{w,F;h,H} = -h^{-1}\big(1\partial - w\partial\ln h1 - \partial HH^{-1} - \partial_H\varpi_{F;H}\big), \quad \text{on } \mathcal{S}_{w,1,F}, \tag{2.21}$$

$$\Delta_{w,F;h,H} = -h^{-1}\big[1\bar\partial\partial - (w\partial\ln h1 + \partial HH^{-1} + \partial_H\varpi_{F;H})\bar\partial\big], \quad \text{on } \mathcal{S}_{w,0,F}, \tag{2.22}$$

$$\begin{aligned}\Delta^\vee_{w,F;h,H} = & -h^{-1}\big[1\bar\partial\partial - \bar\partial\ln h1\partial - (w\partial\ln h1 + \partial HH^{-1} + \partial_H\varpi_{F;H})\bar\partial - w(f_h \\ & -\bar\partial\ln h\partial\ln h)1 + \bar\partial\ln h(\partial HH^{-1} + \partial_H\varpi_{F;H}) - F_H - \bar\partial\partial_H\varpi_{F;H}\big], \quad \text{on } \mathcal{S}_{w,1,F},\end{aligned} \tag{2.23}$$

where $f_h = \bar\partial\partial\ln h$ and $F_H = \bar\partial(\partial HH^{-1})$ are the curvatures of h and H. One can easily check that all these operators map F–valued conformal fields into F–valued conformal fields.

By (2.22) and (2.23), the operators $\Delta_{w,F;h,H}$ and $\Delta^\vee_{w,F;h,H}$ are of the form (2.13). This allows one to obtain the diagonal part of the corresponding heat kernels by applying (2.14). The resulting local expressions are

$$\exp(-t\Delta_{w,F;h,H})_{\rm diag} = \frac{1}{\pi t}h1 + \frac{3w-1}{6\pi}f_h1 + \frac{1}{2\pi}(F_H + \bar\partial\partial_H\varpi_{F;H}) + O(t), \tag{2.24}$$

$$\exp(-t\Delta^\vee_{w,F;h,H})_{\rm diag} = \frac{1}{\pi t}h1 + \frac{2-3w}{6\pi}f_h1 - \frac{1}{2\pi}(F_H + \bar\partial\partial_H\varpi_{F;H}) + O(t). \tag{2.25}$$

Using (2.17), it can be verified that $\exp(-t\Delta_{w,F;h,H})_{\rm diag}$ and $\exp(-t\Delta^\vee_{w,F;h,H})_{\rm diag}$, as given by (2.24) and (2.25), belong to $\mathcal{S}_{1,1,\,\mathrm{End}\,F}$, as expected on general grounds. As they preserve F, one has that $\mathrm{Tr}\big(\exp(-t\Delta^\sharp_{w,F;h,H})Q\big) = \int_\Sigma d^2z\,\mathrm{tr}\big(\varpi_{F;H}\exp(-t\Delta^\sharp_{w,F;h,H})_{\rm diag}Q\big)$ for any bounded self–adjoint multiplicative operator Q in the appropriate Hilbert space corresponding to some element Q of $\mathcal{S}_{0,0,\,\mathrm{End}\,F}$, tr denoting the ordinary fiber trace of $\mathrm{End}\,E$.

Finally one needs to compute the operators $Q_{w,\bar w,F;h,H}$ defined in (2.7). By varying (2.1), one finds the following local expression

$$Q_{w,\bar w,F;h,H} = (1-w-\bar w)\delta\ln h - \delta HH^{-1} + [\delta HH^{-1},\varpi_{F;H}], \quad \text{on } \mathcal{S}_{w,\bar w,F}. \tag{2.26}$$

One can check that this operator preserves F.

Now, one has all the elements required for the study of $\det'_\epsilon\Delta^\sharp_{w,F;h,H}$. In view of physical applications, it turns out to be more convenient to consider a closely related object, the unrenormalized effective action $I^{\rm bare}_{w,F}(h,H;\epsilon)$ defined by

$$I^{\rm bare}_{w,F}(h,H;\epsilon) = \ln\left[\frac{\det'_\epsilon\Delta^\sharp_{w,F;h,H}}{\det M_{w,F;h,H}(\omega)\det M_{w,F;h,H}(\omega^\vee)}\right]. \tag{2.27}$$

Let us find the terms of $I^{\rm bare}_{w,F}(h,H;\epsilon)$, which are singular in the limit $\epsilon \to 0$ corresponding to the removal of the cut–off. By plugging either one of (2.24) and (2.25) into (2.6), one gets

$$I^{\rm bare}_{w,F}(h,H;\epsilon) = -\frac{r_F}{\pi\epsilon}\int_\Sigma d^2z h + K_{w,F}\ln\epsilon + O(1), \tag{2.28}$$

where

$$\begin{aligned} K_{w,F} &= \frac{(3w-1)r_F}{6\pi}\int_\Sigma d^2z f_h + \frac{1}{2\pi}\int_\Sigma d^2z\,\mathrm{tr}\big((F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big) - d_{w,F}, \\ &= \frac{(2-3w)r_F}{6\pi}\int_\Sigma d^2z f_h - \frac{1}{2\pi}\int_\Sigma d^2z\,\mathrm{tr}\big((F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big) - d^\vee_{w,F}. \end{aligned} \tag{2.29}$$

$K_{w,F}$ is independent from (h,H). This can be verified using the variational relations

$$\delta f_h = \bar\partial\partial\delta\ln h, \tag{2.30}$$

$$\delta\,\mathrm{tr}\big((F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big) = \bar\partial\partial\,\mathrm{tr}\big(\delta H H^{-1}\varpi_{F;H}\big), \tag{2.31}$$

the second of which follows from repeated applications of (2.17) and (2.19). By comparing the two expressions of $K_{w,F}$, and recalling that $\mathrm{ind}\,\bar\partial_{w,F} = \dim\ker\bar\partial_{w,F} - \dim\mathrm{coker}\,\bar\partial_{w,F}$, one obtains the index relation

$$\mathrm{ind}\,\bar\partial_{w,F} = \frac{(2w-1)r_F}{2\pi}\int_\Sigma d^2z f_h + \frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}\big((F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big). \tag{2.32}$$

Hence, the right hand side of (2.32) is a topological invariant. Independence from (h,H) follows from (2.30)$-$(2.31). Independence from the background holomorphic structure of E can be verified employing the Beltrami–Koszul parametrization of holomorphic structures described in sect. 3. Indeed, the above integrals are respectively, up to factors, the Gauss–Bonnet invariant and the Chern-Weil invariant of F [23].

Next, let us compute $\delta I^{\rm bare}_{w,F}(h,H;\epsilon)$. By using (2.12), (2.24) – (2.25) and (2.26), one finds

$$\delta I^{\rm bare}_{w,F}(h,H;\epsilon) = -\frac{r_F}{\pi\epsilon}\int_\Sigma d^2z\delta h + A^0_{w,F}(h,H) + O(\epsilon), \tag{2.33}$$

where

$$\begin{aligned} A^0_{w,F}(h,H) &= \frac{(6w^2-6w+1)r_F}{6\pi}\int_\Sigma d^2z\delta\ln h f_h \\ &+ \frac{(2w-1)}{2\pi}\int_\Sigma d^2z\Big[\delta\ln h\,\mathrm{tr}\big((F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big) + \mathrm{tr}\big(\delta H H^{-1}\varpi_{F;H}\big)f_h\Big] \\ &+ \frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}\big(\delta H H^{-1}(F_H+\bar\partial\partial_H\varpi_{F;H})\varpi_{F;H}\big). \end{aligned} \tag{2.34}$$

One may verify that $A^0_{w,F}(h,H)$ is an exact element of $\Omega^1_{\mathfrak{H}_E}$. It clearly must be so, since $\delta I^{\rm bare}_{w,F}(h,H;\epsilon)$ is obviously exact in $\Omega^1_{\mathfrak{H}_E}$. The verification relies on (2.17) and (2.19) and the fact that $H^1(\delta,\Omega^*_{\mathfrak{H}_E})=0$. Remarkably, all three terms in the right hand side of (2.34) are separately exact.

It is possible to integrate (2.33). The integration is carried out along a functional path in $\mathfrak{H}_E$ joining a fiducial reference hermitian structure $(\hat h,\hat H)$ to the hermitian structure considered. The choice of the path is immaterial because of the exactness of the 1–form of $\Omega^1_{\mathfrak{H}_E}$ integrated. By combining (2.28) and (2.33), one obtains

$$I^{\rm bare}_{w,F}(h,H;\epsilon)=-\frac{r_F}{\pi\epsilon}\int_\Sigma d^2zh+K_{w,F}\ln\epsilon+S_{w,F}(h,H;\hat h,\hat H)+s_{w,F}(\hat h,\hat H)+O(\epsilon). \quad (2.35)$$

Here, $s_{w,F}(\hat h,\hat H)$ is a finite functional of $(\hat h,\hat H)$ only and $S_{w,F}(h,H;\hat h,\hat H)$ is formally given by

$$S_{w,F}(h,H;\hat h,\hat H)=\int_{(\hat h,\hat H)}^{(h,H)}A^0_{w,F}(h',H'). \quad (2.36)$$

To perform the functional integration, one introduces the Liouville field ϕ of h relative to $\hat h$ and the Donaldson field Φ of H relative to $\hat H$ [12]. Recall that ϕ and Φ are elements of $\mathcal{S}_{0,0}$ and $\mathcal{S}_{0,0,\,{\rm End}\,E}$, respectively, such that

$$h=\exp\phi\hat h, \quad (2.37)$$

$$\bar\phi=\phi. \quad (2.38)$$

$$H=\exp\Phi\hat H, \quad (2.39)$$

$$H\Phi^\dagger H^{-1}=\Phi. \quad (2.40)$$

Using (2.19), it is straightforward to show that $\varpi_{F;H}$ has a local Taylor expansion in Φ of the form

$$\varpi_{F;H}=\sum_{r=0}^\infty\frac{1}{r!}\varpi^{(r)}_{F;\hat H}(\Phi), \quad (2.41)$$

where, for each $r\geq 0$, $\varpi^{(r)}_{F;\hat H}(\Phi)$ is an element of $\mathcal{S}_{0,0,\,{\rm End}\,E}$ and a homogeneous degree r polynomial in Φ:

$$\begin{aligned}
&\varpi^{(0)}_{F;\hat H}(\Phi)=\varpi_{F;\hat H},\\
&\varpi^{(1)}_{F;\hat H}(\Phi)=-\varpi_{F;\hat H}\Phi(1-\varpi_{F;\hat H}),\\
&\varpi^{(2)}_{F;\hat H}(\Phi)=\varpi_{F;\hat H}\Phi(1-2\varpi_{F;\hat H})\Phi(1-\varpi_{F;\hat H}),\\
&\varpi^{(3)}_{F;\hat H}(\Phi)=\varpi_{F;\hat H}\big[\Phi(3\varpi_{F;\hat H}-1)\Phi(1-\varpi_{F;\hat H})\Phi+\Phi(2-3\varpi_{F;\hat H})\Phi\varpi_{F;\hat H}\Phi\big](1-\varpi_{F;\hat H}),\\
&\qquad\vdots
\end{aligned} \quad (2.42)$$

As functional integration path, it is convenient to use $(g_t, G_t) = (\exp(t\phi)\hat{h}, \exp(t\Phi)\hat{H})$ with $0 \le t \le 1$. One then finds

$$\begin{aligned} S_{w,F}(h,H;\hat{h},\hat{H}) = &-\frac{(6w^2-6w+1)r_F}{6\pi}\int_\Sigma d^2z\Big[\frac{1}{2}\bar{\partial}\phi\partial\phi - f_{\hat{h}}\phi\Big] \\ &-\frac{(2w-1)}{2\pi}\int_\Sigma d^2z\Big[\frac{1}{2}\bar{\partial}\phi\partial\,\mathrm{tr}\,D_{F;\hat{H}}(\Phi)+\frac{1}{2}\partial\phi\bar{\partial}\,\mathrm{tr}\,D_{F;\hat{H}}(\Phi)-f_{\hat{h}}\,\mathrm{tr}\,D_{F;\hat{H}}(\Phi) \\ &\qquad\qquad -\mathrm{tr}\big((F_{\hat{H}}+\bar{\partial}\partial_{\hat{H}}\varpi_{F;\hat{H}})\varpi_{F;\hat{H}}\big)\phi\Big] \\ &-\frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}\Big[\partial_{\hat{H}}\Phi K^*_{F;\hat{H}}(\Phi,\bar{\partial}\Phi)-F_{\hat{H}}D_{F;\hat{H}}(\Phi)+T_{F;\hat{H}}(\Phi)\Big], \end{aligned} \tag{2.43}$$

where

$$D_{F;\hat{H}}(\Phi) = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\sum_{n=0}^{m}\binom{m}{n}\varpi^{(m-n)}_{F;\hat{H}}(\Phi)\Phi\varpi^{(n)}_{F;\hat{H}}(\Phi), \tag{2.44}$$

$$T_{F;\hat{H}}(\Phi) = \sum_{m=0}^{\infty}\frac{1}{(m+1)!}\sum_{n=0}^{m}\binom{m}{n}\partial_{\hat{H}}\varpi^{(m-n)}_{F;\hat{H}}(\Phi)\Phi\bar{\partial}\varpi^{(n)}_{F;\hat{H}}(\Phi), \tag{2.45}$$

$$K^*_{F;\hat{H}}(\Phi,\bar{\partial}\Phi) = \sum_{m=0}^{\infty}\frac{1}{(m+2)!}\sum_{n=0}^{m}\binom{m+1}{n}(-\mathrm{ad}\,\Phi)^{m-n}\sum_{k=0}^{n}\binom{n}{k}\bar{\partial}\big(\varpi^{(n-k)}_{F;\hat{H}}(\Phi)\Phi\big)\varpi^{(k)}_{F;\hat{H}}(\Phi). \tag{2.46}$$

The above expression greatly simplifies when $F = E$, since in such case $\varpi^{(r)}_{F;\hat{H}}(\Phi) = \delta_{r,0}1$. One then finds

$$\begin{aligned} S_{w,F}(h,H;\hat{h},\hat{H}) = &-\frac{(6w^2-6w+1)r_E}{6\pi}\int_\Sigma d^2z\Big[\frac{1}{2}\bar{\partial}\phi\partial\phi - f_{\hat{h}}\phi\Big] \\ &-\frac{(2w-1)}{2\pi}\int_\Sigma d^2z\Big[\frac{1}{2}\bar{\partial}\phi\partial\,\mathrm{tr}\,\Phi+\frac{1}{2}\partial\phi\bar{\partial}\,\mathrm{tr}\,\Phi-f_{\hat{h}}\,\mathrm{tr}\,\Phi-\mathrm{tr}\,(F_{\hat{H}})\phi\Big] \\ &-\frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}\Big[\bar{\partial}\Phi\frac{\exp\mathrm{ad}\,\Phi-1-\mathrm{ad}\,\Phi}{(\mathrm{ad}\,\Phi)^2}\partial_{\hat{H}}\Phi-F_{\hat{H}}\Phi\Big]. \end{aligned} \tag{2.47}$$

Expression (2.43) provides the appropriate generalization of the Liouville action in the present context. By setting $H = \hat{H}$ and $\Phi = 0$ in (2.43), one recovers in fact the customary conformal anomaly. The central charge is

$$c_{w,F} = -2(6w^2-6w+1)r_F \tag{2.48}$$

and is the same as that of r_F copies of a spin w fermionic $b-c$ system. By setting $h = \hat{h}$ and $\phi = 0$, one gets the generalized Weyl anomaly. When $F = E$, so that (2.47) holds, such anomaly is given by the Donaldson action as discussed in ref.[12].

To conclude this section, let us discuss renormalization. From (2.35), it appears that in order to renormalize $I^{\rm bare}_{w,F}(h,H;\epsilon)$, one has to add a counterterm of the form

$$\Delta I^{\rm bare}_{w,F}(h,H;\epsilon)=\lambda_{\rm bare}(\epsilon)\int_\Sigma d^2zh+\nu_{\rm bare}(\epsilon)+\Delta I^{\rm ren}_{w,F}(h,H)+O(\epsilon). \tag{2.49}$$

Here,

$$\lambda_{\rm bare}(\epsilon)=\frac{r_F}{\pi\epsilon}+\lambda_{\rm ren}+O(\epsilon), \tag{2.50}$$

$$\nu_{\rm bare}(\epsilon)=-K_{w,F}\ln\epsilon+\nu_{\rm ren}+O(\epsilon). \tag{2.51}$$

$\Delta I^{\rm ren}_{w,F}(h,H)$ is a finite local functional of (h,H). Its choice defines a renormalization prescription. The renormalized effective action is

$$I^{\rm ren}_{w,F}(h,H)=\lim_{\epsilon\to 0}\Big[I^{\rm bare}_{w,F}(h,H;\epsilon)+\Delta I^{\rm bare}_{w,F}(h,H;\epsilon)\Big]. \tag{2.52}$$

From (2.35) and (2.49), one has

$$I^{\rm ren}_{w,F}(h,H)=\lambda_{\rm ren}\int_\Sigma d^2zh+\nu_{\rm ren}+S_{w,F}(h,H;\hat h,\hat H)+s_{w,F}(\hat h,\hat H)+\Delta I^{\rm ren}_{w,F}(h,H). \tag{2.53}$$

From (2.53) and (2.36), one obtains then

$$\delta I^{\rm ren}_{w,F}(h,H)=\lambda_{\rm ren}\int_\Sigma d^2z\delta h+A_{w,F}(h,H), \tag{2.54}$$

where

$$A_{w,F}(h,H)=A^0_{w,F}(h,H)+\delta\Delta I^{\rm ren}_{w,F}(h,H) \tag{2.55}$$

with $A^0_{w,F}(h,H)$ given by (2.34). (2.54) – (2.55) provide the expression of the generalized Weyl anomaly. Note that the anomaly is local since $\varpi_{F;H}$ is a local functional of H by (2.19).

If minimal subtraction is applied, one has $\lambda_{\rm ren}=\nu_{\rm ren}=0$ and $\Delta I^{\rm ren}_{w,F}(h,H)=0$. Another possibility is to have $\lambda_{\rm ren}\neq 0$. This would lead to a generalization of the Liouville model. Other interesting renormalizations may be considered in specific models, such as the DS ghost system discussed in detail in sect. 4.

Extended conformal field theory, studied in ref. [12], is a particular case of the above framework with $F=E$ and $w=\frac{1}{2}$. The case where $F=E$ but $w\neq\frac{1}{2}$, can be reduced to the latter one by redefining the bundles E and F into $E_w=k^{\otimes w-\frac{1}{2}}\otimes E$ and $F_w=k^{\otimes w-\frac{1}{2}}\otimes F$, respectively, and w into $\frac{1}{2}$ and the hermitian structure (h,H) into $(h,h^{\otimes w-\frac{1}{2}}\otimes H)$.

3. F–special holomorphic structures and F–special automorphisms

Let E be a smooth vector bundle of rank r_E over a compact smooth surface of genus ℓ. Let further F be a subbundle of E of rank $r_F > 0$.

Let $\mathfrak{S}_E$ be the family of holomorphic structures S of E. For $\mathsf{S} \in \mathfrak{S}_E$, let E_{S} be the corresponding holomorphic vector bundle. In general, there does not exist a subbundle F_{S} of E_{S} corresponding to F. The holomorphic structures $\mathsf{S} \in \mathfrak{S}_E$, for which this happens, are called F–special. They form a subfamily $\mathfrak{S}_F$ of $\mathfrak{S}_E$. The holomorphic structures $\mathsf{S} \in \mathfrak{S}_F$ are precisely those, for which the formalism developed in sect. 2 applies.

Let Aut_{cE} and Diff_c be be the groups of smooth automorphisms of E homotopic to id_E and of smooth diffeomorphisms of Σ homotopic to id_Σ, respectively. If $\alpha \in \mathrm{Aut}_{cE}$, there exists $f_\alpha \in \mathrm{Diff}_c$ such that $\pi \circ \alpha = f_\alpha \circ \pi$ and that $\alpha|_{E_p}$ is a linear isomorphisms of E_p onto $E_{f_\alpha(p)}$ for every $p \in \Sigma$, where π is the bundle projection and E_p is the fiber of E at p. In general, for a given $\alpha \in \mathrm{Aut}_{cE}$, $\alpha|_{E_p}$ does not map F_p onto $F_{f_\alpha(p)}$, *i. e.* α does not respect the subbundle F. The automorphisms $\alpha \in \mathrm{Aut}_{cE}$, for which this happens, are called F–special. They form a subgroup Aut_{cF} of Aut_{cE}. This is the relevant symmetry group for the field theoretic constructions of sect. 2.

There is a natural action of Aut_{cE} on $\mathfrak{S}_E$. This associates to any $\mathsf{S} \in \mathfrak{S}_E$ and any $\alpha \in \mathrm{Aut}_{cE}$ the pull back $\alpha^*\mathsf{S} \in \mathfrak{S}_E$ of S by α (see [12] for a detailed discussion). A simple but important theorem states that Aut_{cF} preserves $\mathfrak{S}_F$.

Proof. Let $\mathsf{S} \in \mathfrak{S}_E$. S is a collection of trivializations $\{(z_{\mathsf{S}a}, u_{\mathsf{S}a})\}$, where $z_{\mathsf{S}a}$ is a complex coordinate on Σ, $u_{\mathsf{S}a}$ is a fiber coordinate and, whenever defined, $\bar\partial_{\mathsf{S}a} z_{\mathsf{S}b} = 0$ and $u_{\mathsf{S}a} = T_{\mathsf{S}ab} \circ \pi u_{\mathsf{S}b}$ with $T_{\mathsf{S}ab}$ an $r_E \times r_E$ matrix valued function such that $\bar\partial_{\mathsf{S}c} T_{\mathsf{S}ab} = 0$. If $\alpha \in \mathrm{Aut}_{cE}$ and $\mathsf{S} \in \mathfrak{S}_E$, then $\alpha^*\mathsf{S} \in \mathfrak{S}_E$ with $z_{\alpha^*\mathsf{S}a} = z_{\mathsf{S}b} \circ f_\alpha$ and $u_{\alpha^*\mathsf{S}a} = u_{\mathsf{S}b} \circ \alpha$ and $T_{\alpha^*\mathsf{S}ac} = T_{\mathsf{S}bd} \circ f_\alpha$ for suitably related a, c and b, d. If $\mathsf{S} \in \mathfrak{S}_F$, there exists, for each trivialization $(z_{\mathsf{S}a}, u_{\mathsf{S}a})$, an $r_E \times r_E$ matrix valued function $\Theta_{\mathsf{S}a}$ such that $(\Theta_{\mathsf{S}a} \circ \pi u_{\mathsf{S}a})(F \cap \pi^{-1}(\mathrm{dom}\, z_{\mathsf{S}a}))$ has the last $r_E - r_F$ components identically zero, $\Theta_{\mathsf{S}a} T_{\mathsf{S}ab} {\Theta_{\mathsf{S}b}}^{-1}$ has vanishing lower left $(r_E - r_F) \times r_F$ block and $\bar\partial_{\mathsf{S}a} \Theta_{\mathsf{S}a} = 0$. If $\alpha \in \mathrm{Aut}_{cF}$ and $\mathsf{S} \in \mathfrak{S}_F$, then it is straightforward to verify that $\alpha^*\mathsf{S} \in \mathfrak{S}_F$ by setting $\Theta_{\alpha^*\mathsf{S}a} = \Theta_{\mathsf{S}b} \circ f_\alpha$. *QED*

Let $\mathsf{S} \in \mathfrak{S}_E$ be a holomorphic structure of E and $(h_{\mathsf{S}}, H_{\mathsf{S}})$ be a hermitian structure on E_{S}. If $\alpha \in \mathrm{Aut}_{cE}$, then the pull–back $(\alpha^* h_{\alpha^*\mathsf{S}}, \alpha^* H_{\alpha^*\mathsf{S}})$ of $(h_{\mathsf{S}}, H_{\mathsf{S}})$ by α is a hermitian structure on $E_{\alpha^*\mathsf{S}}$.

Let $\mathcal{H}_{w,\bar w,F;h,H;\mathsf{S}}$ be the Hilbert space defined in sect. 1 with the holomorphic structure $\mathsf{S} \in \mathfrak{S}_F$ indicated. If $\alpha \in \mathrm{Aut}_{cF}$, then the pull-back operator α^* is a unitary operator of $\mathcal{H}_{w,\bar w,F;h,H;\mathsf{S}}$ onto $\mathcal{H}_{w,\bar w,F;\alpha^*h,\alpha^*H;\alpha^*\mathsf{S}}$

Proof. This follows easily from (2.1) using the relations $\alpha^*\phi_{a\alpha^*\mathsf{S}} = \phi_{b\mathsf{S}} \circ f_\alpha$ with $\phi_\mathsf{S} \in \mathcal{H}_{w,\bar{w},F;h,H;\mathsf{S}}$ and $\alpha^* h_{a\alpha^*\mathsf{S}} = (h_{b\mathsf{S}} \circ f_\alpha$ and $\alpha^* H_{a\alpha^*\mathsf{S}} = H_{b\mathsf{S}} \circ f_\alpha)$ for suitably related a and b. *QED*

If $\mathsf{S} \in \mathfrak{S}_F$ and $\alpha \in \mathrm{Aut}_{cF}$, then $\bar{\partial}_{w,F;\alpha^*\mathsf{S}} = \alpha^* \circ \bar{\partial}_{w,F;\mathsf{S}} \circ \alpha^{*-1}$. This implies, among other things, that $\Delta^\sharp_{w,F;\alpha^*h,\alpha^*H;\alpha^*\mathsf{S}} = \alpha^* \circ \Delta^\sharp_{w,F;h,H;\mathsf{S}} \circ \alpha^{*-1}$. α^* being unitary, the spectrum of $\Delta^\sharp_{w,F;h,H;\mathsf{S}}$ is Aut_{cF} invariant.

The above geometrical treatment is elegant but abstract. One would like to translate it into the language of field theory, which is the one suitable for physical applications. This can be achieved as follows [12].

For any pair of holomorphic structures S_1, S_2, there exist two distinguished sections $\lambda_{\mathsf{S}_1\mathsf{S}_2}$ and $V_{\mathsf{S}_1\mathsf{S}_2}$ of $k_{\mathsf{S}_1} \otimes k_{\mathsf{S}_1}{}^{\otimes -1}$ and $E_{\mathsf{S}_1} \otimes E_{\mathsf{S}_2}{}^\vee$, respectively, called intertwiners. Write a generic trivialization of S_i as $(z_{\mathsf{S}_i}, u_{\mathsf{S}_i})$, where z_{S_i} is a complex coordinate on Σ and u_{S_i} is a fiber coordinate. Then, $\lambda_{\mathsf{S}_1\mathsf{S}_2} = \partial_{\mathsf{S}_1} z_{\mathsf{S}_2}$ and $V_{\mathsf{S}_1\mathsf{S}_2}$ is defined by the relation $u_{\mathsf{S}_1} = V_{\mathsf{S}_1\mathsf{S}_2} \circ \pi u_{\mathsf{S}_2}$. The intertwiners define an isomorphism between the space of sections of each vector bundle constructed by means of k_{S_1} and E_{S_1} and the space of sections of the corresponding bundle constructed by means of k_{S_2} and E_{S_2}. Hence, the field content of a field theory having E as topological background is described completely by the spaces of sections of vector bundles built by means of k_{S_0} and E_{S_0} for a fiducial reference holomorphic structure S_0. All relevant field theoretic relations may be thus written in terms of the trivializations of S_0. By convention, when a field or a combination of fields carries no subscript S, then it is represented in terms of S_0. Note that by E and k it is denoted both the holomorphic vector bundle E_{S_0} and canonical line bundle k_{S_0} and their smooth counterparts. This generates no confusion since from the context it will be clear which is meant.

There is a one–to–one correspondence between the family $\mathfrak{S}_E$ of holomorphic structures S of E and the family of pairs (μ, A^*_A), where μ is a Beltrami field and A^*_A is a Koszul field [12]. Recall that a Beltrami field μ is an element of $\mathcal{S}_{-1,1}$ such that $\sup_\Sigma |\mu| < 1$ and that a Koszul field A^*_A is an element of $\mathcal{S}_{0,1,\mathrm{End}\,E}$. For $\mathsf{S} = (\mu, A^*_A) \in \mathfrak{S}_E$, one has

$$\mu = \bar{\partial} z_\mathsf{S} / \partial z_\mathsf{S}, \tag{3.1}$$

$$A^*_A = (\bar{\partial} - \mu\partial) V_\mathsf{S} V_\mathsf{S}{}^{-1} + \mu A, \tag{3.2}$$

where A is a fixed $(1,0)$ connection of E, $\bar{\partial} \equiv \bar{\partial}_{\mathsf{S}_0}$ and $V_\mathsf{S} \equiv V_{\mathsf{S}_0\mathsf{S}}$ [12].

All the identities of sect. 1, valid for an arbitrary holomorphic structure $\mathsf{S} \in \mathfrak{S}_F$, may be easily written in the Beltrami–Koszul parametrization by performing the formal substitutions

$$d^2z \to d^2z(1-\bar{\mu}\mu), \tag{3.3}$$

$$\bar{\partial} \to \frac{1}{(1-\bar{\mu}\mu)}(\bar{\partial}-\mu\partial-w\partial\mu), \quad \text{on } \mathcal{S}_{w,0}, \tag{3.4}$$

$$\bar{\partial} \to \frac{1}{(1-\bar{\mu}\mu)}\big(\bar{\partial}-\mu\partial_H-w\partial\mu-\operatorname{ad}A_H^*\big), \quad \text{on } \mathcal{S}_{w,0,\,\mathrm{End}\,E}, \tag{3.5}$$

$$f_h \to \frac{1}{(1-\bar{\mu}\mu)}\bigg[f_h-\big(\partial-\bar{\mu}\bar{\partial}-\bar{\partial}\bar{\mu}\big)\frac{\partial_h\mu}{1-\bar{\mu}\mu}-\big(\bar{\partial}-\mu\partial-\partial\mu\big)\frac{\bar{\partial}_h\bar{\mu}}{1-\bar{\mu}\mu}\bigg], \tag{3.6}$$

$$F_H \to \frac{1}{(1-\bar{\mu}\mu)}\bigg[F_H-\big(\partial_H-\bar{\mu}\bar{\partial}-\bar{\partial}\bar{\mu}\big)\frac{A_H^*}{1-\bar{\mu}\mu}-\big(\bar{\partial}-\mu\partial_H-\partial\mu\big)\frac{HA_H^{*\dagger}H^{-1}}{1-\bar{\mu}\mu} + \frac{[A_H^*,HA_H^{*\dagger}H^{-1}]}{1-\bar{\mu}\mu}\bigg]. \tag{3.7}$$

Here, $\partial_h = \partial + \partial \ln h$ is the covariant derivative associated to the metric connection $\partial \ln h$ acting on $\mathcal{S}_{-1,1}$. A_H^* is given by (3.2) with $A = \partial H H^{-1}$. By using the Beltrami–Koszul parametrization one may also check that the the integral expression (2.32) is independent from the holomorphic structure chosen, as expected from the index theorem.

The Beltrami–Koszul parametrization allows one to state a condition for a holomorphic structure $\mathsf{S} \in \mathfrak{S}_E$ to be F–special. $\mathsf{S} \in \mathfrak{S}_F$ if and only if

$$\big(\bar{\partial}\varpi_{F;H}\varpi_{F;H}\big)_{\mathsf{S}} = 0. \tag{3.8}$$

Proof. This condition is necessary, as explained in sect. 2. It is also sufficient. For if (3.8) holds, there exists on each trivialization domain a local holomorphic frame in E_{S} spanning F, implying that E_{S} contains a holomorphic subbundle F_{S} corresponding to F. *QED*
The dependence of this condition on the metric H is only apparent. In fact, using (2.19) it is easy to show that if S satisfies (3.8) for a given hermitian metric H, then it does also satisfy it for any other close metric H'. It must clearly be so, for the space $\mathfrak{S}_F$, by its definition, does not depend on a choice of hermitian structure. (3.8) can be written in the Beltrami–Koszul parametrization, where it reads

$$\big(\bar{\partial}-\mu\partial_A-\operatorname{ad}A_A^*\big)\varpi_{F;H}\varpi_{F;H} = 0, \tag{3.9}$$

where $\partial_A = \partial - \mathrm{ad}\, A$ is the covariant derivative on $\mathrm{End}\, E$ associated to A. This is the field theoretic constraint that must be obeyed by a holomorphic structure $\mathsf{S} = (\mu, A^*_A)$ in order it to belong to $\mathfrak{S}_F$.

In the analysis of symmetries, it is much simpler to proceed at the infinitesimal level. Let s be the nilpotent Slavnov operator, $s^2 = 0$. Let c and m be the automorphisms ghosts [12]. c is the diffeomorphism ghost associated to the natural map $\mathrm{Aut}_{cE} \to \mathrm{Diff}_c$ defined earlier. c is a section of k^{-1} valued in $\bigwedge^1(\mathrm{Lie}\,\mathrm{Aut}_{cE})^\vee$. M corresponds to the action of Aut_{cE} on the fibers of E. For a given background $(1,0)$ connection of E, $M - cA$ is a section of $\mathrm{End}\, E$ valued in $\bigwedge^1(\mathrm{Lie}\,\mathrm{Aut}_{cE})^\vee$. The Maurer–Cartan equations of Aut_{cE} yield

$$sc = (c\partial + \bar{c}\bar{\partial})c, \tag{3.10}$$

$$sM = (c\partial + \bar{c}\bar{\partial})M - \frac{1}{2}[M, M] \tag{3.11}$$

[12]. The action of Aut_{cE} on $\mathfrak{S}_E$ induces an action on the Beltrami–Koszul fields (μ, A^*_A) given by

$$s\mu = \big(\bar{\partial} - \mu\partial + \partial\mu\big)C, \tag{3.12}$$

$$sA^*_A = \big(\bar{\partial} - \mu\partial_A - \mathrm{ad}\, A^*_A\big)X_A + C\big(\partial_A A^*_A - \bar{\partial}A\big), \tag{3.13}$$

where

$$C = c + \mu\bar{c}, \tag{3.14}$$

$$X_A = cA + \bar{c}A^*_A - M. \tag{3.15}$$

C and X_A are sections of k^{-1} and $\mathrm{End}\, E$ valued in $\bigwedge^1(\mathrm{Lie}\,\mathrm{Aut}_{cE})^\vee$ and depending on (μ, A^*_A), respectively. Further

$$sC = C\partial C, \tag{3.16}$$

$$sX_A = C\partial_A X_A + \frac{1}{2}[X_A, X_A]. \tag{3.17}$$

The pull-back action of Aut_{cE} on the space $\mathfrak{H}_E$ of hermitian structures (h, H) of E yields

$$s \ln h = (c\partial + \bar{c}\bar{\partial}) \ln h + \partial c + \mu\partial\bar{c} + \bar{\partial}\bar{c} + \bar{\mu}\bar{\partial}c, \tag{3.18}$$

$$sHH^{-1} = (c\partial + \bar{c}\bar{\partial})HH^{-1} - M - HM^\dagger H^{-1}. \tag{3.19}$$

The action of the F–special automorphism group Aut_{cF} can still be expressed at the infinitesimal level by means of the automorphism ghost fields c and M. The restriction to $\mathrm{Lie}\,\mathrm{Aut}_{cF}$ shows up as a relation obeyed by c and M, which will be derived in a moment.

It is not difficult to show that an automorphism $\alpha \in \mathrm{Aut}_{cE}$ belongs to Aut_{cF} if and only if

$$\varpi_{F;\alpha^* H} = \alpha^* \varpi_{F;H}. \tag{3.20}$$

Proof. Denote by (z_a, u_a) the trivializations of the reference holomorphic structure, as done earlier. If $\alpha \in \mathrm{Aut}_{cE}$, then, for any two trivializations (z_a, u_a) and (z_b, u_b) such that $\operatorname{dom} z_a \cap f_\alpha(\operatorname{dom} z_b) \neq \emptyset$, there exists a local $r_E \times r_E$ smooth matrix function $\hat{\alpha}_{ab}$ such that $u_a \circ \alpha = \hat{\alpha}_{ab} \circ \pi u_b$. One further has $\alpha^* \Theta_b = \hat{\alpha}_{ab}{}^{-1} \Theta_a \circ f_\alpha \hat{\alpha}_{ab}$ for any element Θ of $\mathcal{S}_{0,0,\operatorname{End} E}$. Let $\alpha \in \mathrm{Aut}_{cF}$. Then, for any $x \in F$, one has $(\varpi_{F;H} \circ \pi u)(\alpha(x)) = u(\alpha(x))$, since $\alpha(x) \in F$. This implies that, for any $x \in F$, $(\alpha^* \varpi_{F;H} \circ \pi u)(x) = u(x)$. Then, since $\alpha^* \varpi_{F;H}$ is $\alpha^* H$–hermitian, (3.20) holds. Next, let $\alpha \in \mathrm{Aut}_{cE}$ satisfy (3.20). Then, since $(\varpi_{F;\alpha^* H} \circ \pi u)(x) = u(x)$ for any $x \in F$, one has that $(\alpha^* \varpi_{F;H} \circ \pi u)(x) = u(x)$ for $x \in F$. This implies that for any $x \in F$, one has $(\varpi_{F;H} \circ \pi u)(\alpha(x)) = u(\alpha(x))$. Hence, for any $x \in F$, $\alpha(x) \in F$, so that $\alpha \in \mathrm{Aut}_{cF}$. *QED*

Using (2.19), one can also show that this condition is actually independent from the metric H, as expected on general grounds form the metric independence of Aut_{cF}. Going over the infinitesimal formulation and using (2.19) and (3.19), one finds that, for the F–special symmetry,

$$\big(c\partial + \bar{c}\bar{\partial} - \operatorname{ad} M\big)\varpi_{F;H}\varpi_{F;H} = 0, \tag{3.21}$$

which is the constraint on c and M looked for. If (μ, A^*_A) is an F–special holomorphic structure, so that (3.9) is fulfilled, then (3.21) can be stated in terms of the (μ, A^*_A) dependent ghost fields C and X_A given by (3.14) and (3.15) as follows

$$(C\partial_A + \operatorname{ad} X_A)\varpi_{F;H}\varpi_{F;H} = 0. \tag{3.22}$$

It can be verified that (3.21) is compatible with (3.10) and (3.11) in the following sense. If one applies s to the left hand side of (3.21) and uses (3.10), (3.11) and (2.19), one obtains a result that is linear in the left hand side of (3.21). Thus, enforcing the constraint (3.21) is compatible with the action Aut_{cE}. This is expected on general grounds and verified here. Similarly, if one applies s to the left hand side of (3.9) and uses (3.12), (3.13) and (2.19), one obtains an expression linear in the left hand sides of (3.9) and (3.22). Hence, imposing the constraints (3.9) and (3.22) is again compatible with the action of Aut_{cE}.

In the Beltrami–Koszul parametrization of holomorphic structures the determinant of $\Delta^\sharp_{w,F;h,H;\mathsf{S}}$ and the associated bare and renormalized effective actions become functionals

of the geometrical fields μ, $\bar{\mu}$, A^*_H and $A^*_H{}^\dagger$. The Aut_{cF} invariance of the spectrum of $\Delta^\sharp_{w,F;h,H;\mathsf{S}}$ implies that its determinant also is invariant. Hence

$$s\,{\det}'_\epsilon\Delta^\sharp_{w,F;h,H;\mu,\bar{\mu},A^*_H,A^*_H{}^\dagger} = 0. \tag{3.23}$$

The bare effective action $I^{\mathrm{bare}}_{w,F}(h,H;\mathsf{S};\epsilon)$ cannot really be considered a functional over the space $\mathfrak{H}_E\times\mathfrak{S}_F$ because of the ambiguity inherent in the choice of the bases of zero modes. For this reason, $I^{\mathrm{bare}}_{w,F}(h,H;\mathsf{S};\epsilon)$ is invariant under Aut_{cF} only up to redefinitions of the zero mode bases. However, the exponential of $I^{\mathrm{bare}}_{w,F}(h,H;\mathsf{S};\epsilon)$ can be viewed as a section of a line bundle on $\mathfrak{H}_E\times\mathfrak{S}_F$. As such, $I^{\mathrm{bare}}_{w,F}(h,H;\mathsf{S};\epsilon)$ is in fact Aut_{cF} invariant and one has

$$sI^{\mathrm{bare}}_{w,F}(h,H;\mu,\bar{\mu},A^*_H,A^*_H{}^\dagger;\epsilon) = 0. \tag{3.24}$$

The exponential of the renormalized effective action $I^{\mathrm{ren}}_{w,F}(h,H;\mathsf{S})$ may be viewed similarly as a section of the same line bundle on $\mathfrak{H}_E\times\mathfrak{S}_F$. The counterterm $\Delta I^{\mathrm{bare}}_{w,F}(h,H;\mathsf{S};\epsilon)$ given by $(2,49)$ is Aut_{cF} invariant if $\Delta I^{\mathrm{ren}}_{w,F}(h,H;\mathsf{S})$ is. In that case case, $I^{\mathrm{ren}}_{w,F}(h,H;\mathsf{S})$, also, is Aut_{cF} invariant and one has

$$sI^{\mathrm{ren}}_{w,F}(h,H;\mu,\bar{\mu},A^*_H,A^*_H{}^\dagger) = 0. \tag{3.25}$$

4. The Drinfeld–Sokolov ghost system

The basic algebraic data entering in the definition of the model are the following: *i*) a simple complex Lie group G; *ii*) an $SL(2,\mathbb{C})$ subgroup S of G invariant under the compact conjugation $\dagger$ of G. Let t_{-1}, t_0, t_{+1} be a set of standard generators of $\mathfrak{s}$, *i. e.*

$$[t_{+1},t_{-1}] = 2t_0, \quad [t_0,t_{\pm 1}] = \pm t_{\pm 1}, \tag{4.1}$$

$$t_d{}^\dagger = t_{-d}, \quad d = -1,0,+1. \tag{4.2}$$

To the Cartan element t_0 of $\mathfrak{s}$, there is associated a halfinteger grading of $\mathfrak{g}$: the subspace $\mathfrak{g}_m$ of $\mathfrak{g}$ of degree $m\in\mathbb{Z}/2$ is the eigenspace of $\mathrm{ad}\,t_0$ with eigenvalue m. One can further define a bilinear form χ on $\mathfrak{g}$ by $\chi(x,y) = \mathrm{tr}_{\,\mathrm{ad}}(t_{+1}[x,y])$, $x,y\in\mathfrak{g}$ [14], where $\mathrm{tr}_{\,\mathrm{ad}}$ denotes the Cartan–Killing form. The restriction of χ to $\mathfrak{g}_{-\frac{1}{2}}$ is non singular. By Darboux theorem, there is a direct sum decomposition $\mathfrak{g}_{-\frac{1}{2}} = \mathfrak{p}_{-\frac{1}{2}}\oplus\mathfrak{q}_{-\frac{1}{2}}$ of $\mathfrak{g}_{-\frac{1}{2}}$ into subspaces of the same dimension, which are maximally isotropic and dual to each other with respect to χ. Set

$$\mathfrak{x} = \mathfrak{p}_{-\frac{1}{2}}\oplus\bigoplus_{m\le -1}\mathfrak{g}_m. \tag{4.3}$$

$\mathfrak{x}$ is a negative graded nilpotent subalgebra of $\mathfrak{g}$.

On a Riemann surface Σ of genus ℓ with a spinor structure $k^{\otimes\frac{1}{2}}$, one may define the G valued holomorphic 1–cocycle

$$L_{ab} = \exp(-\ln k_{ab} t_0) \exp(\partial_a k_{ab}{}^{-1} t_{-1}). \tag{4.4}$$

This in turn defines a holomorphic principal G–bundle, the Drinfeld–Sokolov (DS) bundle [16,24]. $\mathrm{Ad}\,L$ is one of the associated holomorphic vector bundles. The $\mathfrak{x}$–valued sections of $\mathrm{Ad}\,L$ span a subbundle $\mathrm{Ad}\,L_{\mathfrak{x}}$ of $\mathrm{Ad}\,L$, since $\mathfrak{x}$ is invariant under $\mathrm{ad}\,t_0$ and $\mathrm{ad}\,t_{-1}$.

The DS ghost system $\beta-\gamma$, described in the introduction, is governed by the action (1.1), where β and γ are anticommuting sections of $k\otimes \mathrm{Ad}\,L$ and $\mathrm{Ad}\,L$ valued in $\mathfrak{g}/\mathfrak{x}^\perp$ and $\mathfrak{x}$, respectively, $\mathfrak{x}^\perp$ being the orthogonal complement of $\mathfrak{x}$ with respect to $\mathrm{tr}_{\,\mathrm{ad}}$. The effective action of the DS ghost system is thus of the type described in sect. 2 with $E = \mathrm{Ad}\,L$, $F = \mathrm{Ad}\,L_{\mathfrak{x}}$ and $w = 0$. The hermitian structures of $\mathrm{Ad}\,L$ considered here are of the form $(h, \mathrm{Ad}\,H)$, where H is a hermitian metric of L. From (2.54), (2.55) and (2.34), one finds that the renormalized effective action $I^{\mathrm{ren}}_{DS}(h,H)$ satisfies the Weyl anomalous Ward identity

$$\delta I^{\mathrm{ren}}_{DS}(h,H) = \lambda_{\mathrm{ren}} \int_\Sigma d^2 z \delta h + A_{DS}(h,H), \tag{4.5}$$

$$A_{DS}(h,H) = A^0_{DS}(h,H) + \delta\Delta I^{\mathrm{ren}}_{DS}(h,H), \tag{4.6}$$

where

$$\begin{aligned} A^0_{DS}(h,H) &= \frac{r_{DS}}{6\pi}\int_\Sigma d^2 z \delta \ln h f_h \\ &- \frac{1}{2\pi}\int_\Sigma d^2 z\Big[\delta \ln h \,\mathrm{tr}\big((\mathrm{ad}\,F_H + \bar\partial\partial_H \varpi_H)\varpi_H\big) + \mathrm{tr}\big(\mathrm{ad}\,(\delta H H^{-1})\varpi_H\big) f_h\Big] \\ &+ \frac{1}{\pi}\int_\Sigma d^2 z\, \mathrm{tr}\big(\mathrm{ad}\,(\delta H H^{-1})(\mathrm{ad}\,F_H + \bar\partial\partial_H\varpi_H)\varpi_H\big), \end{aligned} \tag{4.7}$$

with $r_{DS} = \dim \mathfrak{x}$ and $\Delta I^{\mathrm{ren}}_{DS}(h,H)$ is a local functional of (h,H) [2].

For the DS principal bundle L, there exists a distinguished choice of the fiber metric H for any given hermitian metric h on the base Σ, namely

$$H_h = \exp(-\partial \ln h t_{-1}) \exp(-\ln h t_0) \exp(-\bar\partial \ln h t_{+1}). \tag{4.8}$$

It is not difficult to show that the corresponding projector ϖ_{H_h} is given by

$$\varpi_{H_h} = \exp(-\partial \ln h \,\mathrm{ad}\, t_{-1}) p_{\mathfrak{x}} \exp(\partial \ln h \,\mathrm{ad}\, t_{-1}), \tag{4.9}$$

[2] In this section, I shall suppress the indices w and F to lighten the notation.

where $p_{\mathfrak{x}}$ is the orthogonal projector of $\mathfrak{g}$ onto $\mathfrak{x}$ with respect to the hermitian inner product on $\mathfrak{g}$ defined by $(x,y) = \mathrm{tr}_{\,\mathrm{ad}}(x^\dagger y)$ for $x, y \in \mathfrak{g}$.

For the DS ghost system, besides the minimal subtraction renormalization prescription, corresponding to setting $\Delta I^{\mathrm{ren}}_{DS}(h,H) = 0$, there is another relevant renormalization defined by the choice

$$\Delta I^{\mathrm{ren}}_{DS}(h,H) = \frac{1}{2\pi}\int_\Sigma d^2z\Big[\int_{H_h}^{H} \mathrm{tr}\big(\mathrm{ad}\,(\delta H' H'^{-1})\varpi_{H'}\big)\Big] f_h. \tag{4.10}$$

The functional 1–form $\mathrm{tr}\big(\mathrm{ad}\,(\delta H H^{-1})\varpi_H\big)$ of $\Omega^1_{\mathfrak{H}_L}$ is exact. Thus, the above functional line integral does not depend on the choice of the functional path joining H_h to H in $\mathfrak{H}_L$. Using the Taylor expansion (2.41) and (4.8) and (4.9), one can verify that $\Delta I^{\mathrm{ren}}_{DS}(h,H)$ is a local functional of (h,H). Further, using (2.31), one can show that

$$\delta\Delta I^{\mathrm{ren}}_{DS}(h,H) = \frac{1}{2\pi}\int_\Sigma d^2z\Big[\delta\ln h\,\mathrm{tr}\big((\mathrm{ad}\,F_H + \bar\partial\partial_H\varpi_H)\varpi_H\big) + \mathrm{tr}\big(\mathrm{ad}\,(\delta H H^{-1})\varpi_H\big)f_h\Big]$$
$$- \frac{1}{2\pi}\int_\Sigma d^2z\Big[\delta\ln h\,\mathrm{tr}\big((\mathrm{ad}\,F_{H_h} + \bar\partial\partial_{H_h}\varpi_{H_h})\varpi_{H_h}\big) + \mathrm{tr}\big(\mathrm{ad}\,(\delta H_h {H_h}^{-1})\varpi_{H_h}\big)f_h\Big]. \tag{4.11}$$

Hence, on account of (4.5) – (4.7), choosing $\Delta I^{\mathrm{ren}}_{DS}(h,H)$ to be given by (4.10), one obtains a renormalized effective action $I^{\mathrm{ren}}_{DS}(h,H)$, for which the classical H equations are of the form

$$\mathrm{tr}\big(\mathrm{ad}\,(\delta H H^{-1})(\mathrm{ad}\,F_H + \bar\partial\partial_H\varpi_H)\varpi_H\big) + \ldots = 0. \tag{4.12}$$

The ellipses denote terms coming from the matter sector of the model, which will not be discussed here [15]. The relevant point is that the above classical equations, including the contributions coming from the matter sector not shown, do not contain the surface metric h. Thus, the classical H dynamics induced by $I^{\mathrm{ren}}_{DS}(h,H)$ is conformally invariant.

With the metric H_h available, one may define the reduced renormalized effective action

$$I^{\mathrm{ren}}_{DS}(h) = I^{\mathrm{ren}}_{DS}(h, H_h) \tag{4.13}$$

for any choice of the renormalization prescription. Here, $\Delta I^{\mathrm{ren}}_{DS}(h,H)$ is meaningfully chosen to be of the form

$$\Delta I^{\mathrm{ren}}_{DS}(h,H) = \frac{\kappa_0}{\pi}\int_\Sigma d^2z h^{-1} {f_h}^2, \tag{4.14}$$

where κ_0 is a real constant. By using (4.5) – (4.7), one can obtain the Weyl anomalous Ward identity obeyed by $I^{\mathrm{ren}}_{DS}(h)$. This can be written in rather explicit form, because of

the simple dependence of H_h and ϖ_{H_h} on h. By a somewhat lengthy but straightforward calculation, one finds

$$\delta I^{\rm ren}_{DS}(h) = -\frac{c_{DS}}{12\pi}\int_\Sigma d^2z\delta\ln hf_h + \frac{\kappa_0-\kappa_{DS}}{\pi}\delta\int_\Sigma d^2zh^{-1}{f_h}^2, \tag{4.15}$$

where

$$c_{DS} = -2\,{\rm tr}\big[\big(6({\rm ad}\,t_0)^2+6\,{\rm ad}\,t_0+1\big)p_{\mathfrak{x}}\big], \tag{4.16}$$

$$\kappa_{DS} = {\rm tr}\big({\rm ad}\,t_{+1}\,{\rm ad}\,t_{-1}p_{\mathfrak{x}}\big). \tag{4.17}$$

Choosing $\kappa_0=\kappa_{DS}$ yields a renormalized effective action $I^{\rm ren}_{DS}(h)$ describing a conformal field theory of central charge c_{DS}. This is precisely the central charge of the DS ghost system as computed with the methods of hamiltonian reduction and conformal field theory [14] [3]. For a generic value of κ_0, one obtains a renormalized effective action with a $\int\sqrt{h}{R_h}^2$ term yielding a model of induced $2d$ gravity of the same type as that considered in refs. [18–19].

It is possible to compute the index of the ghost kinetic operator $\bar\partial$ in the above framework. One uses the general relation (2.32) and carries out the calculation using the convenient fiber metric H_h. The result is

$$\begin{aligned}{\rm ind}\,\bar\partial =& -\frac{r_{DS}}{2\pi}\int_\Sigma d^2zf_h + \frac{1}{\pi}\int_\Sigma d^2z\,{\rm tr}\big(({\rm ad}\,F_{H_h}+\bar\partial\partial_{H_h}\varpi_{H_h})\varpi_{H_h}\big)\\ =& -\,{\rm tr}\big[\big(2\,{\rm ad}\,t_0+1\big)p_{\mathfrak{x}}\big](\ell-1).\end{aligned} \tag{4.18}$$

The dimension of the kernel of $\bar\partial$ is the number of linearly independent γ–zero modes. It can be computed as follows. Recall that to any linearly independent generator of $\mathfrak{g}$ of t_0 degree $-m<0$ there correspond d_m linearly independent holomorphic sections of ${\rm Ad}\,L$, where d_m is the dimension of space $\mathcal{S}^{\rm hol}_{m,0}$ of holomorphic elements of $\mathcal{S}_{m,0}$ [16]. Recall also that $d_1=\ell$ and that $d_m=(2m-1)(\ell-1)$ for $m\geq\frac{3}{2}$ and $\ell\geq2$ [17]. Using these remarks and (4.3), one finds that

$$\dim\ker\bar\partial = \dim\mathfrak{g}_{-1}+\frac{1}{2}\dim\mathfrak{g}_{-\frac{1}{2}}d_{\frac{1}{2}} - {\rm tr}\big[\big(2\,{\rm ad}\,t_0+1\big)p_{\mathfrak{x}}\big](\ell-1), \quad \ell\geq2. \tag{4.19}$$

The dimension of the cokernel of $\bar\partial$ is the number of linearly independent β–zero modes. This can be easily computed using (4.18) and (4.19). One finds

$$\dim{\rm coker}\,\bar\partial = \dim\mathfrak{g}_{-1}+\frac{1}{2}\dim\mathfrak{g}_{-\frac{1}{2}}d_{\frac{1}{2}}, \quad \ell\geq2. \tag{4.20}$$

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[3] The odd looking sign of the mid term in the right hand side of (4.16) is due to the fact that $\mathfrak{x}$ is negative graded.

The above analysis has been carried out for a fixed $\mathrm{Ad}\,L_{\mathfrak{x}}$–special holomorphic structure of the smooth vector bundle $\mathrm{Ad}\,L$ characterized by the holomorphic G–valued 1–cocycle (4.4). One may take such holomorphic structure as a reference one. Let us now study the family of $\mathrm{Ad}\,L_{\mathfrak{x}}$–special holomorphic structure of $\mathrm{Ad}\,L$.

Let R be a holomorphic projective connection. Then

$$A_R = \frac{1}{2}t_{+1} - Rt_{-1} \tag{4.21}$$

is a holomorphic $(1,0)$ connection of L. Below, A_R will be used as background. All fields built using A_R will carry a subscript R.

For any Beltrami field μ, consider the holomorphic structure $\mathsf{S}_\mu = (\mu, \mathrm{ad}\,A^*_R(\mu))$ whose Koszul field $A^*_R(\mu)$ is of the form

$$A^*_R(\mu) = \frac{1}{2}\mu t_{+1} - \partial\mu t_0 - (\partial^2 + R)\mu t_{-1} \tag{4.22}$$

[4]. It is straightforward to verify that $A^*_R(\mu)$ belongs to $\mathcal{S}_{0,1,\,\mathrm{Ad}\,L}$, so that $A^*_R(\mu)$ is a *bona fide* Koszul field. A generic holomorphic structure $\mathsf{S} = (\mu, \mathrm{ad}\,A^*_R)$ of $\mathrm{Ad}\,L$ can be written in the form

$$A^*_R = A^*_R(\mu) + a^*, \tag{4.23}$$

where a^* is some element of $\mathcal{S}_{0,1,\,\mathrm{Ad}\,L}$. Let $\mathfrak{S}_{DS}$ be the family of all holomorphic structures $\mathsf{S} = (\mu, \mathrm{ad}\,A^*_R)$ such that a^* is $\mathfrak{x}$–valued. Then $\mathfrak{S}_{DS} \subset \mathfrak{S}_{\mathrm{Ad}\,L_{\mathfrak{x}}}$, *i. e.* $\mathfrak{S}_{DS}$ consists of $\mathrm{Ad}\,L_{\mathfrak{x}}$–special holomorphic structures.

Proof. To begin with, one notes that, for the DS bundle, one has

$$\bar\partial\varpi_H\varpi_H = 0, \quad \partial\varpi_H\varpi_H = 0. \tag{4.24}$$

The first relation is just (2.17). For a general vector bundle, the second relation would not be covariant. However, here, because of the specific form of the cocycle (4.4) and the fact that $\mathfrak{x}$ is invariant under $\mathrm{ad}\,t_0$ and $\mathrm{ad}\,t_{-1}$, it actually is. (4.24) is shown as follows. Let L_0 be the holomorphic G–valued 1–cocycle defined by $L_{0ab} = \exp(-\ln k_{ab} t_0)$. A generic metric H of L undergoes a Gauss type factorization of the form

$$H = KH_0K^\dagger, \tag{4.25}$$

[4] Strictly speaking, the Koszul field is $\mathrm{ad}\,A^*_R$. However, in this section, I shall use this name for the field A^*_R itself.

where K is an $\exp\mathfrak{x}$–valued section of $L\otimes L_0^\vee$ and H_0 is some metric of L_0 valued in $\exp\mathfrak{k}_0$ with $\mathfrak{k}_0=\mathfrak{q}_{-\frac{1}{2}}\oplus\mathfrak{g}_0\oplus\mathfrak{q}_{-\frac{1}{2}}{}^\dagger$. Next, pick a basis $\{e_\xi|\xi\in I\}$ of $\mathfrak{x}$ constituted by eigenvectors of $\operatorname{ad}t_0$. Then, one has

$$\begin{aligned}\varpi_H&=\operatorname{Ad}K\sum_{\xi,\eta\in I}e_\xi\otimes g(H_0)^{-1}{}_{\xi\eta}\tilde{e}_\eta\operatorname{Ad}H_0{}^{-1}\operatorname{Ad}K^{-1},\\ g(H_0)_{\xi\eta}&=\tilde{e}_\xi(\operatorname{Ad}H_0{}^{-1}e_\eta),\end{aligned}\tag{4.26}$$

where $\tilde{e}_\eta=\operatorname{tr}_{\mathrm{ad}}(e_\eta^\dagger\,\cdot)$. From this expression, it is not difficult to check the validity of (4.24). Using (4.21), (4.23) and (4.24) and the fact that $\mathfrak{x}$ is invariant under $\operatorname{ad}t_0$ and $\operatorname{ad}t_{-1}$, one finds that $\mathsf{S}=(\mu,\operatorname{ad}A_R^*)$ fulfills (3.9) when a^* is $\mathfrak{x}$–valued, so that S is special. *QED*

Note that $\mathfrak{S}_{DS}$ is strictly contained in $\mathfrak{S}_{\mathrm{Ad}\,L_{\mathfrak{x}}}$. For instance, if $\omega^*\in\mathcal{S}_{0,1}$ and γ is a $(1,0)$ connection of the line bundle k, then, setting $a^*=\omega^*t_0-\gamma\omega^*t_{-1}$, the holomorphic structure $(\mu,\operatorname{ad}A_R^*)$ defined by (4.23) is special but it is not contained in $\mathfrak{S}_{DS}$. In W–algebras, $\mathfrak{S}_{DS}$ is the relevant class of special holomorphic structures since the constraint on the Wess–Zumino current is implemented at the lagrangian level by coupling it to a $\mathfrak{x}$–valued gauge field, namely a^* [14–15].

Next, consider the automorphism ghosts c and M. Set

$$M(c,\bar{c},\mu)=\big(\partial c+\mu\partial\bar{c}\big)t_0+\big(\partial(\partial c+\mu\partial\bar{c})+\partial\mu\partial\bar{c}\big)t_{-1}.\tag{4.27}$$

One can verify that $M(c,\bar{c},\mu)-cA_R$ is a section of $\operatorname{Ad}L$ valued in $\bigwedge^1(\operatorname{Lie}\operatorname{Aut}_{cL})^\vee$. Using (3.10) and (3.12), one verifies that $M(c,\bar{c},\mu)$ fulfills (3.11). Write

$$M=M(c,\bar{c},\mu)+m\tag{4.28}$$

with m a section of $\operatorname{Ad}L$ valued in $\bigwedge^1(\operatorname{Lie}\operatorname{Aut}_{cL})^\vee$. Since M and $M(c,\bar{c},\mu)$ both satisfy (3.11), one has

$$sm=\big(c\partial+\bar{c}\bar{\partial}-\operatorname{ad}M(c,\bar{c},\mu)\big)m-\frac{1}{2}[m,m].\tag{4.29}$$

Now, it is easily checked that (c,M) fulfills the specialty condition (3.21) if m is $\mathfrak{x}$–valued. Such constraint defines a subgroup Aut_{cDS} of $\operatorname{Aut}_{c\,\mathrm{Ad}\,L_{\mathfrak{x}}}$.

Proof. It follows trivially from (4.24) that (c,M) fulfills the specialty condition (3.21) once the $\mathfrak{x}$–valuedness of m is enforced. Note that $\mathfrak{x}$–valuedness of m is respected by (4.29), since $\mathfrak{x}$ is a subalgebra of $\mathfrak{g}$ invariant under $\operatorname{ad}t_0$ and $\operatorname{ad}t_{-1}$. Hence, the constraint defines a subgroup Aut_{cDS} of $\operatorname{Aut}_{c\,\mathrm{Ad}\,L_{\mathfrak{x}}}$. *QED*

Reasoning in the same way as at the end of the previous paragraph, one can see that Aut_{cDS} is strictly contained in $\mathrm{Aut}_{c\,\mathrm{Ad}\,L_{\mathfrak{x}}}$. In W–gravity, however, the relevant symmetry group is Aut_{cDS} since the renormalized matter effective action is invariant only under Aut_{cDS} when the background holomorphic structures S are constrained to belong to $\mathfrak{S}_{DS}$ [14–15].

Following (3.15), one defines

$$X_R(C) = cA_R + \bar{c}A_R^*(\mu) - M(c, \bar{c}, \mu). \tag{4.30}$$

As suggested by the notation, $X_R(C)$ depends on c, $\bar{c}$ and μ through the combination C defined in (3.14). In fact

$$X_R(C) = \frac{1}{2}Ct_{+1} - \partial Ct_0 - (\partial^2 + R)Ct_{-1}. \tag{4.31}$$

Remarkably, $A_R^*(\mu)$ fulfills (3.13) with X_A replaced by $X_R(C)$ and $X_R(C)$ fulfills (3.17). It follows from (3.15), (4.23) and (4.28) that

$$X_R = X_R(C) + x, \tag{4.32}$$

where x is a section of $\mathrm{Ad}\,L$ valued in $\bigwedge^1(\mathrm{Lie}\,\mathrm{Aut}_{cL})^\vee$. Explicitly, from (3.15), (4.23), (4.28) and (4.30), one has

$$x = \bar{c}a^* - m. \tag{4.33}$$

Using the fact that A_R^* and $A_R^*(\mu)$ both obey (3.13) with the appropriate ghost field X_R and that X_R and $X_R(C)$ both obey (3.17), one finds the relations

$$sa^* = \big(C\partial_R + \mathrm{ad}\,X_R(C)\big)a^* + \big(\bar{\partial} - \mu\partial_R - \mathrm{ad}\,A_R^*(\mu) - \mathrm{ad}\,a^*\big)x, \tag{4.34}$$

$$sx = \big(C\partial_R + \mathrm{ad}\,X_R(C)\big)x + \frac{1}{2}[x, x], \tag{4.35}$$

where $\partial_R = \partial_{A_R}$. If a^* is $\mathfrak{x}$–valued, so that the corresponding holomorphic structure is special, then (C, X_R) fulfills the specialty condition (3.22) if x is $\mathfrak{x}$–valued. Note that (4.34) and (4.35) respect $\mathfrak{x}$–valuedness.

In the Beltrami–Koszul parametrization, restricting to holomorphic structures $\mathsf{S} \in \mathfrak{S}_{DS}$, the DS ghost action reads

$$S_{DS}(\beta, \beta^\dagger, \gamma, \gamma^\dagger; \mu, \bar{\mu}, a^*, a^{*\dagger}) = \frac{1}{\pi}\int_\Sigma d^2z\,\mathrm{tr}_{\mathrm{ad}}\big[\beta\big(\bar{\partial} - \mu\partial_R - \mathrm{ad}\,A_R^*(\mu) - \mathrm{ad}\,a^*\big)\gamma\big] + \mathrm{c.\ c.}, \tag{4.36}$$

where a^* is $\mathfrak{x}$–valued. Using this expression, it is straightforward to compute the classical energy–momentum tensor $T_{DS}(\beta,\gamma)$. One has

$$T_{DS}(\beta,\gamma) = \pi\frac{\delta S_{DS}}{\delta\mu}(\beta,\beta^\dagger,\gamma,\gamma^\dagger;0,0,0,0) = \mathrm{tr}_{\mathrm{ad}}\big(\partial_R\gamma\beta + D_R[\gamma,\beta]\big),$$
$$D_R = \frac{1}{2}t_{+1} + t_0\partial - t_{-1}(\partial^2 + R). \tag{4.37}$$

Similarly, one can compute the classical gauge current $J_{DS}(\beta,\gamma)$. One finds

$$J_{DS}(\beta,\gamma) = \pi\frac{\delta S_{DS}}{\delta a^*}(\beta,\beta^\dagger,\gamma,\gamma^\dagger;0,0,0,0) = [\gamma,\beta]. \tag{4.38}$$

Note that $T_{DS}(\beta,\gamma)$ contains a second derivative improvement term $\mathrm{tr}_{\mathrm{ad}}\big(D_R J_{DS}(\beta,\gamma)\big)$, a common feature in W–algebras. Note also that $J_{DS}(\beta,\gamma)$ is valued in $\mathfrak{g}/[\mathfrak{x},\mathfrak{x}^\perp]$ since β is $\mathfrak{g}/\mathfrak{x}^\perp$–valued and γ is $\mathfrak{x}$–valued.

In the above geometrical formulation, I have not defined a notion of stability for special holomorphic structures $\mathsf{S} = (\mu, \mathrm{ad}\,A_R^*) \in \mathfrak{S}_{DS}$ with a fixed Beltrami field μ. In the analysis below, it will be assumed that the holomorphic structure on Σ defined by μ is generic in the sense that $d_{\frac{1}{2}} = 0, 1$ depending on whether the spinor structure is even or odd, respectively. Now, no structure $\mathsf{S} \in \mathfrak{S}_{DS}$ is stable in the customary sense. Indeed, the space $\mathcal{S}^{\mathrm{hol}}_{0,0\,\mathrm{Ad}\,L;\mathsf{S}}$ of holomorphic elements in $\mathcal{S}_{0,0\,\mathrm{Ad}\,L:\mathsf{S}}$ is non trivial, while, for stable structures, $\mathcal{S}^{\mathrm{hol}}_{0,0\,\mathrm{Ad}\,L;\mathsf{S}}$ must vanish [17]. In physical terms, $\mathcal{S}^{\mathrm{hol}}_{0,0\,\mathrm{Ad}\,L;\mathsf{S}}$ is the space of the holomorphic infinitesimal gauge transformations of $\mathrm{Ad}\,L_{\mathsf{S}}$ and, for stable structures S, has minimal dimension. Here, the relevant symmetry group is the DS gauge group Gau_{cDS}, which is the gauge subgroup of Aut_{cDS} [5]. So, one may define stability as follows. S is said stable if the space $\mathcal{S}^{\mathrm{hol}}_{0,0\,\mathrm{Ad}\,L_{\mathfrak{x}};\mathsf{S}}$ of holomorphic $\mathfrak{x}$–valued elements in $\mathcal{S}_{0,0\,\mathrm{Ad}\,L;\mathsf{S}}$ has minimal dimension. Let us denote by $\mathfrak{S}^{\mathrm{stab}}_{DS}$ the subspace of $\mathfrak{S}_{DS}$ of all stable holomorphic structures S of $\mathfrak{S}_{DS}$. Clearly, $\mathfrak{S}^{\mathrm{stab}}_{DS}$ is preserved by Aut_{cDS}. Non stable holomorphic structures must satisfy in the Beltrami–Koszul parametrization certain linear conditions. They thus span a submanifold of $\mathfrak{S}_{DS}$ of finite codimension. Hence, $\mathfrak{S}^{\mathrm{stab}}_{DS}$ is dense in $\mathfrak{S}_{DS}$.

In W–gravity there are two geometrical structures of crucial importance in analogy to string theory. The first is the holomorphic subgroup $\mathrm{Gau}^{\mathrm{hol}}_{cDS;\mathsf{S}}$ of the DS gauge group Gau_{cDS} for any stable holomorphic structure $\mathsf{S} \in \mathfrak{S}^{\mathrm{stab}}_{DS}$. The second is DS Teichmueller

[5] In geometrical terms, an element α of the automorphism group Aut_{cE} of a vector bundle E is a gauge transformation if the induced diffeomorphism $f_\alpha = \mathrm{id}_\Sigma$.

space $\mathrm{Teich}_{DS} = \mathfrak{S}_{DS}^{\mathrm{stab}}/\mathrm{Gau}_{cDS}$ of stable holomorphic structures $\mathsf{S} \in \mathfrak{S}_{DS}^{\mathrm{stab}}$ modulo Gau_{cDS}. Their dimensions can be computed. By direct calculation, one finds that

$$\dim \mathrm{Gau}_{cDS;\mathsf{S}}^{\mathrm{hol}} = \dim \mathfrak{g}_{-1} + n_* d_{\frac{1}{2}} - \mathrm{tr}\left[\left(2\,\mathrm{ad}\,t_0 + 1\right)p_{\mathfrak{r}}\right](\ell - 1), \quad \ell \geq 2, \tag{4.39}$$

where $n_* = \min_{x \in \mathfrak{p}_{-\frac{1}{2}}} \dim \ker \mathrm{ad}\,x|_{\mathfrak{p}_{-\frac{1}{2}}}$. Clearly, n_* depends on $\mathfrak{s}$ and $n_* \geq 1$. Using the index relation (4.18) and (4.39), one finds that

$$\dim \mathrm{Teich}_{DS} = \dim \mathfrak{g}_{-1} + n_* d_{\frac{1}{2}}, \quad \ell \geq 2. \tag{4.40}$$

The calculation of these numbers is one of the main results of this paper.

Proof. Consider the holomorphic structure $\mathsf{S}_\mu = (\mu, \mathrm{ad}\,A_R^*(\mu))$ defined earlier. It is not difficult to check that the intertwiner V_{S_μ} of S_μ is given by $\exp(-\ln \partial z_{\mathsf{S}_\mu} t_0)\exp(\partial(\partial z_{\mathsf{S}_\mu})^{-1}t_{-1})$ and that $L_{\mathsf{S}_\mu ab} = \exp(-\ln k_{\mathsf{S}_\mu ab} t_0)\exp(\partial_a k_{\mathsf{S}_\mu ab}{}^{-1} t_{-1})$. Note that this 1–cocycle is of the DS form (4.4). Hence, choosing the reference holomorphic structure of $\mathrm{Ad}\,L$, so that the induced holomorphic structure on Σ is generic in the sense stated above, one can assume that $\mu = 0$ without loss of generality. The holomorphic structures $\mathsf{S} \in \mathfrak{S}_{DS}$, in which one is interested, are therefore of the form $(0, a^*)$ with a^* an $\mathfrak{r}$–valued element of $\mathcal{S}_{0,1,\mathrm{Ad}\,L}$. Let Θ be a section of $\bar{k}^{\otimes \bar{w}} \otimes \mathrm{Ad}\,L$. One can decompose Θ as follows

$$\Theta = \sum_{m \in \mathbb{Z}/2, |m| \leq j_*} \Theta^{(m)} \quad \text{with} \quad [\mathrm{ad}\,t_0, \Theta^{(m)}] = m\Theta^{(m)}, \tag{4.41}$$

where j_* is the highest eigenvalue of $\mathrm{ad}\,t_0$. Applying theorems 3.2 and 3.3 of ref. [16], one can easily show the following. If $\Theta^{(m)} = 0$ for $p < m \leq j_*$ with $-j_* \leq p < j_*$, then $\Theta^{(p)}$ is a section of $k^{\otimes -p} \otimes \bar{k}^{\otimes \bar{w}} \otimes \mathfrak{g}_p$. Pick a holomorphic projective connection R. For any section θ of $k^{\otimes -p} \otimes \bar{k}^{\otimes \bar{w}} \otimes \mathfrak{g}_p$ with $-j_* \leq p \leq -\frac{1}{2}$, there exists a section $T_R(\theta)$ of $\bar{k}^{\otimes \bar{w}} \otimes \mathrm{Ad}\,L$ such that

$$T_R(\theta)^{(m)} = 0, \quad \text{for } p < m \leq j_*, \quad T_R(\theta)^{(p)} = \theta. \tag{4.42}$$

Further, when $\bar{w} = 0$, one has

$$\bar{\partial} T_R(\theta) = T_R(\bar{\partial}\theta). \tag{4.43}$$

Consider the equation

$$\left(\bar{\partial} - \mathrm{ad}\,a^*\right)\eta = 0 \tag{4.44}$$

with η an $\mathfrak{r}$–valued section of $\mathrm{Ad}\,L$. The space of solution of this equation is precisely $\ker \bar{\partial}_{\mathsf{S}} \cong \mathrm{Lie}\,\mathrm{Gau}_{cDS;\mathsf{S}}^{\mathrm{hol}}$. Now, set $\eta_0 = \eta$. Then, by (4.3), ${\eta_0}^{(m)} = 0$ for $m > -\frac{1}{2}$, so that ${\eta_0}^{(-\frac{1}{2})}$ is a section of $k^{\otimes \frac{1}{2}} \otimes \mathfrak{g}_{-\frac{1}{2}}$, as recalled above. It follows from (4.44) and the fact

that $a^{*(m)}=0$ for $m>-\frac{1}{2}$ that $\bar\partial{\eta_0}^{(-\frac{1}{2})}=0$ by grading reasons. There are $d_{\frac{1}{2}}\dim\mathfrak{g}_{-\frac{1}{2}}/2$ linearly independent such ${\eta_0}^{(-\frac{1}{2})}$. Define $\eta_1=\eta_0-T_R\big({\eta_0}^{(-\frac{1}{2})}\big)$. By (4.42), one has that ${\eta_1}^{(m)}=0$ for $m>-1$, so that ${\eta_1}^{(-1)}$ is a section of $k\otimes\mathfrak{g}_1$. By (4.43), the holomorphicity of ${\eta_0}^{(-\frac{1}{2})}$, t_0–grading reasons and (4.44) one has further that $\bar\partial{\eta_1}^{(-1)}=[a^{*(-\frac{1}{2})},{\eta_0}^{(-\frac{1}{2})}]$. The general solution of this equation, if it exists, is a linear inhomogeneous function of $d_1\dim\mathfrak{g}_{-1}$ complex parameters since ${\eta_1}^{(-1)}$ is determined up to the addition of an arbitrary section $\zeta^{(-1)}$ of $k\otimes\mathfrak{g}_1$ such that $\bar\partial\zeta^{(-1)}=0$. A solution exists provided the integrability condition $\int_\Sigma d^2z[a^{*(-\frac{1}{2})},{\eta_0}^{(-\frac{1}{2})}]=0$ is satisfied. Since $d_{\frac{1}{2}}=0,1$, ${\eta_0}^{(-\frac{1}{2})}$ is of the form $\sigma x^{(-\frac{1}{2})}$, where σ is a holomorphic section of $k^{\otimes\frac{1}{2}}$ such that $\sigma\neq0$ if $d_{\frac{1}{2}}=1$ and $x^{(-\frac{1}{2})}\in\mathfrak{p}_{-\frac{1}{2}}$. Hence, the integrability condition reduces into $[\int_\Sigma d^2z\sigma a^{*(-\frac{1}{2})},x^{(-\frac{1}{2})}]=0$. If a^* is to represent a stable holomorphic structure, this must be a condition constraining $x^{(-\frac{1}{2})}$ only. From here, it is easy to see that, for a stable holomorphic structure, the space of allowed ${\eta_0}^{(-\frac{1}{2})}$ has dimension $n_*d_{\frac{1}{2}}$. Next, define $\eta_2=\eta_1-T_R\big({\eta_1}^{(-1)}\big)$. By (4.42), one has that ${\eta_2}^{(m)}=0$ for $m>-\frac{3}{2}$, so that ${\eta_2}^{(-\frac{3}{2})}$ is a section of $k^{\otimes\frac{3}{2}}\otimes\mathfrak{g}_{-\frac{3}{2}}$. By (4.43), one has further that $\bar\partial{\eta_2}^{(-\frac{3}{2})}=\big(\bar\partial\eta_1-T_R\big(\bar\partial{\eta_1}^{(-1)}\big)\big)^{(-\frac{3}{2})}$. The general solution of this equation always exists and is a linear inhomogeneous function of $d_{\frac{3}{2}}\dim\mathfrak{g}_{-\frac{3}{2}}$ complex parameters, since ${\eta_2}^{(-\frac{3}{2})}$ is determined up to the addition of an arbitrary section $\zeta^{(-\frac{3}{2})}$ of $k^{\otimes\frac{3}{2}}\otimes\mathfrak{g}_{-\frac{3}{2}}$ such that $\bar\partial\zeta^{(-\frac{3}{2})}=0$. The procedure can now be iterated. At the p–th step one defines a section ${\eta_p}^{(-\frac{p+1}{2})}$ of $k^{\otimes\frac{p+1}{2}}\otimes\mathfrak{g}_{-\frac{p+1}{2}}$ satisfying an equation whose general solution is a linear inhomogeneous function of $d_{\frac{p+1}{2}}\dim\mathfrak{g}_{-\frac{p+1}{2}}$ complex parameters. In conclusion, for a stable structure, $\dim\ker\bar\partial_{\mathsf{S}}=n_*d_{\frac{1}{2}}+\sum_{p\geq1}d_{\frac{p+1}{2}}\dim\mathfrak{g}_{-\frac{p+1}{2}}$. Using that $d_1=\ell$ and that $d_m=(2m-1)(\ell-1)$ for $m\geq\frac{3}{2}$ and $\ell\geq2$ [17] and the remark just below (4.44), one obtains (4.39) readily. To compute $\dim\mathrm{Teich}_{DS}$, one notes $\dim\mathrm{Teich}_{DS}=\dim\mathrm{Gau}^{\mathrm{hol}}_{cDS;\mathsf{S}}-\mathrm{ind}\bar\partial_{\mathsf{S}}$ by a reasoning analogous to that used to compute the dimension of the ordinary Teichmueller space in string theory. Then, (4.40) follows immediately from the index relation (4.18) and (4.39). *QED*

Acknowledgements. I wish to voice my gratitude to M. Matone, E. Aldrovandi, G. Velo and expecially M. Bauer and R. Stora for helpful discussions.

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